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Going with the Flow: CHALLENGING STUDENTS TO MAKE ASSUMPTIONS

Future middle school teachers tested the waters of modeling in the classroom with a bath versus shower water conservation problem.

Many current and future teachers have little experience with modeling and how to integrate it into their teaching. However, with the introduction of the Common Core State Standards for Mathematics (CCSSM) and its emphasis on mathematical modeling in all grades (CCSSI 2010), this integration has become paramount. Therefore, middle-grades teachers must work to lay the groundwork for modeling, which must then continue into high school.

In this article, we describe a unit designed to introduce modeling to prospective

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teachers (referred to in this article as *students*) and consider how this unit can illuminate classroom practice. We focus on one problem, the Water Conservation task, which is also well suited for use with middle school students. Because the reasoning and judgment demanded by the modeling process applies at all grade levels, we believe that this type of experience can further the process of mathematical modeling in the middle school classroom. Through teaching students and sharing our experiences, we wish to bring a clearer understanding of mathematical modeling to professional developers; practicing teachers; and, ultimately, students in the classroom.

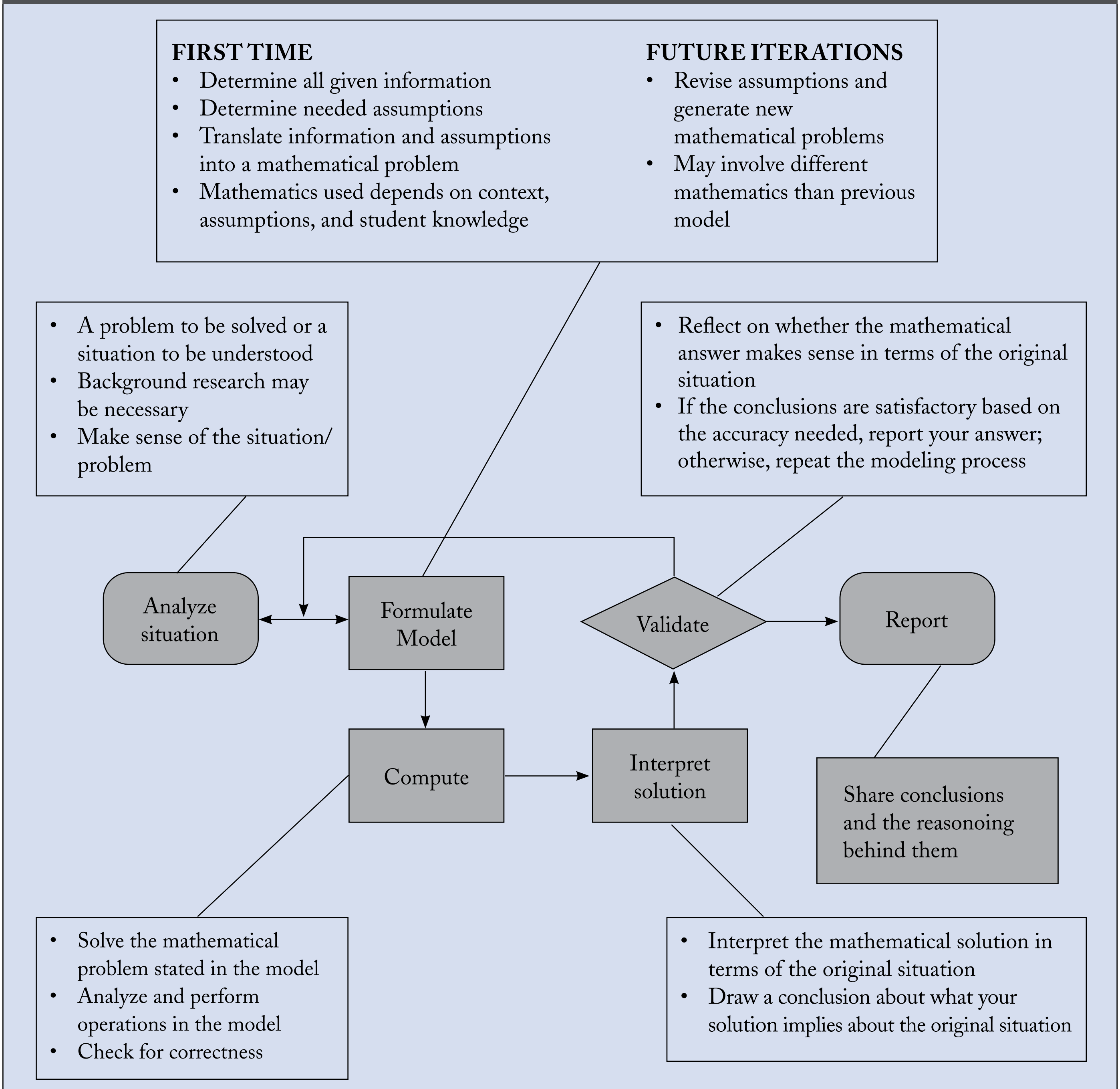
OVERVIEW OF THE MODELING UNIT

In this unit, we wanted to (1) advance the students' understanding of the modeling process, keeping in mind what will be expected of them as future teachers; and (2) better understand the evolution of students' views of modeling to provide information for future work in teacher education and professional development.

The three-week unit was designed for an introductory mathematics education course for preservice secondary teachers. Before and after the unit, we gave the sixteen students in the course a questionnaire asking them to define modeling, to explain what it would look like in middle school and high school, and to consider its relationship to "word problems" and to "real life." The unit involved a series of three mathematical tasks: The Car Wash problem (NCTM 2005, p. 17), The Cell Phone problem (Anhalt and Cortez 2015), and the Water Conservation problem (Anhalt 2014). Along with each problem, students reflected on the modeling cycle in relation to the tasks. In designing the tasks, our goal was to focus on the role of making assumptions and how the assumptions affect the model created.

Modeling helps learners develop habits of mind. When they are connecting mathematics to real-world situations, learners must consider the assumptions being made about

Fig. 1 The modeling cycle is both a multistep and an iterative process.



Source: Adapted from CCSSI (2010)

the specific context and how these assumptions can be translated into mathematics. Modeling also provides opportunities to engage in other mathematical practices, such as persevering in problem solving, developing precision in language and mathematical models, and making mathematical arguments (Koestler et al. 2013). In this article, our focus on the Water Conservation task considers how students can engage in the process of modeling and how it can be implemented in the classroom.

WHAT IS MATHEMATICAL MODELING?

Several interpretations of the word *model* are described in *Principles and Standards for School Mathematics* (NCTM 2000) and occur in mathematics education. Two views were commonly held by the students. Several students referred to "physical materials with which students work in school—manipulative models" (NCTM 2000, p. 70). For example, one student commented that "modeling can be done in a middle or high school in different ways. For instance using block tiles to solve simple algebraic equations." Other students interpreted modeling "as if it were roughly synonymous with representation" (NCTM 2000, 70); for instance, stating that modeling involved "using graphs, charts, physical models, equations, and many other things to represent the mathematical content they are learning."

Neither of these views represents mathematical modeling as defined in CCSSM or *Principles and Standards*. The views above involve starting with a mathematical concept and representing it in multiple ways, such as with physical manipulatives, graphs, or pictures. Mathematical modeling generally runs in the opposite direction: One begins with a real-world phenomenon, determines how mathematical concepts can be used to understand the phenomenon, and returns to the original phenomenon. Modeling involves a multistep and iterative process, called the *modeling cycle*, which is illustrated in **figure 1**.

Another major theme in the students' views—that modeling involves real-world problems—showed indications of this perspective, with one student writing that modeling involved using "mathematical concepts to look at, interpret, and solve real-life, tangible problems." However, despite considering the use of mathematics for understanding real-world problems, the students primarily gave examples of relatively simple "application" problems that would not require making and revisiting assumptions, which is a central feature of the modeling process. Although modeling could be misinterpreted as an "application" problem to be done only after a mathematical concept has been taught, it is important to recognize that modeling can also be one way for students to learn new mathematics (Lesh and Harel 2003; Koestler et al. 2013).

Fig. 2 Analyzing flow rates, in part, required students to work through the modeling process.

Water Conservation Task

Some water conservationists say that showering uses less water than bathing. Others say that this is not true! Keep in mind that older showerheads have a flow rate of up to 3.4 gallons/minute whereas energy-saving showerheads have a flow rate as low as 1.9 gallons/minute. Bathtubs also vary in size. Provide a method to determine if a shower or a bath uses more water and explain your approach.

THE WATER CONSERVATION TASK

The Water Conservation task (see **fig. 2**) involves finding a method for determining whether taking a shower or bath uses more water. The problem requires learners to identify needed information and to fill in this information through online research or by making reasonable assumptions. This problem was assigned as homework, and students were instructed to use appropriate resources and to come to class prepared to discuss their work. However, we encourage teachers to use this task in the classroom, so that they can provide greater support to students throughout the modeling process.

Analyzing the Situation and Making Assumptions

We have learned to be explicit about the purpose of modeling. For instance, in this task we wish to explain clearly that a model may be developed to help consumers make decisions about whether to shower or bathe. The reason for this decision is that an environmental group may want to create public awareness about saving water or an apartment

What Does CCSSM Say about Modeling?

Mathematical modeling is both the fourth Standard for Mathematical Practice, intended to span all grade levels (CCSSI 2010, p. 7), and a conceptual category in the high school grades (pp. 72–73). Modeling involves the following:

1. Making sense of a situation
2. Determining given and needed information, making assumptions, and translating these into mathematics, often in the form of a problem to be solved
3. Computing a solution to the problem
4. Interpreting the results in the context of the original situation
5. Determining if the results are reasonable and helpful and then either returning to steps 1 or 2 or
6. Reporting the solution

Fig. 3 Both arithmetic and algebraic strategies were used with the Water Conservation task.

	Compare Shower with Bath	Solve for Length of Shower
Arithmetic	Calculate the amount of water used, assuming 10 minute and 20 minute showers for each showerhead. 1.9 (gal./min.) × 10 min. = 19 gal. 3.4 (gal./min.) × 10 min. = 34 gal. 1.9 (gal./min.) × 20 min. = 38 gal. 3.4 (gal./min.) × 20 min. = 68 gal. A standard-size bathtub holds 42 gallons of water. The shower uses less than this in every case, except the last one.	Find the amount of time when each showerhead uses more water than filling the bathtub, assuming an average-size bathtub holds 58 gallons. 58 gal. ÷ 3.4 gal./min. ≈ 17 min. 58 gal. ÷ 1.9 gal./min. ≈ 31 min. Shorter showers than these times use less water than a bath.
Algebraic	This method of comparing water usage would work for people with different data. Bath usage = volume of 3/4 tub converted to gallons Shower usage = showerhead flow rate × average shower time	Depending on the size of the bathtub and the flow rate of the showerhead, find the number of minutes that would equalize the shower's water usage with the bathtub size.

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complex manager may need to decide what kinds of showerheads to install. Clarifying these goals helps guide learners as they engage in the modeling process. Therefore, we recommend having a class discussion about the possible reasons for engaging in this task and selecting one reason before moving forward. In addition, although this context is likely familiar to most students, they should have an opportunity to ask clarifying questions to ensure that they have enough background knowledge before moving on. Prompts such as "What information do you need to determine how much water is used during a shower?" and "What do you need to know to figure out how much water a bath consumes?" can guide students in unpacking the context.

Students may need to obtain several pieces of information to make reasonable assumptions. First, the water capacity of a bathtub must be researched. Our students all found appropriate information, ranging from 42 gallons to 70 gallons, mostly through online research, although one student found the number experimentally by measuring her bathtub. Some students used the bathtub capacity as the amount of water used in a bath, whereas others used the more realistic assumption that a person would only fill the tub to about 3/4 capacity. Leaving these decisions to the students requires them to consider which assumptions they will incorporate into their models.

Second, all the students used the proportional relationship of

(gallons of water) = (flow rate) × (length of shower)

when calculating shower water usage. This relationship carried the unstated assumption that the flow rate was constant. In addition, almost all the students explicitly or implicitly assumed that only two flow rates were possible for showers (the two extremes given in the task), instead of considering a range. Depending on the model used, students may need to estimate the length of a shower by making a reasonable guess; doing research online; or collecting data, such as by timing family members.

Third, it is important to note that there were also a few cases in which the students' stated "assumptions" were either irrelevant to their mathematical model (e.g., "I would assume people would rather have energy-saving showerheads.") or were not actually assumptions (e.g., "To save more energy, take less time in the shower and buy newer showerheads!").

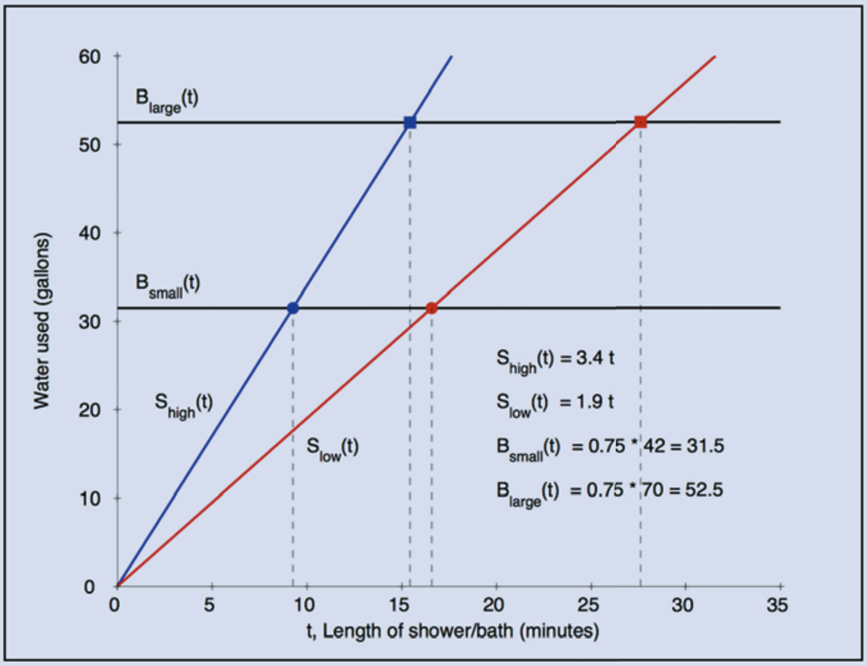
Model Formulation and Computation

The students had two major strategies for approaching the task. They either compared the water used in a shower to that used in a bath or solved for the break-even length of time for a shower. Within each of these two strategies, some students took an arithmetic approach focused on one or more specific cases; others took an algebraic approach by describing a general method that could apply to a range of situations. **Figure 3** shows examples of strategies that are summarized from several of the students' work.

Interpretation and Validation

All students correctly interpreted their mathematical solutions in terms of water conservation of a bath versus a shower. In the validation step, the students were satisfied with their results and did not see a need to reformulate their model. The results of the arithmetic approach could have been deemed unsatisfactory because they address only a limited number of cases; this could have led to the students revising their assumptions and making a second iteration through the modeling cycle where a more general (algebraic) approach could be used. However, none of the students did this.

Fig. 4 The graph shows the best-case and worst-case scenarios for showers (*S*) and baths (*B*), given the assumptions identified by the students.

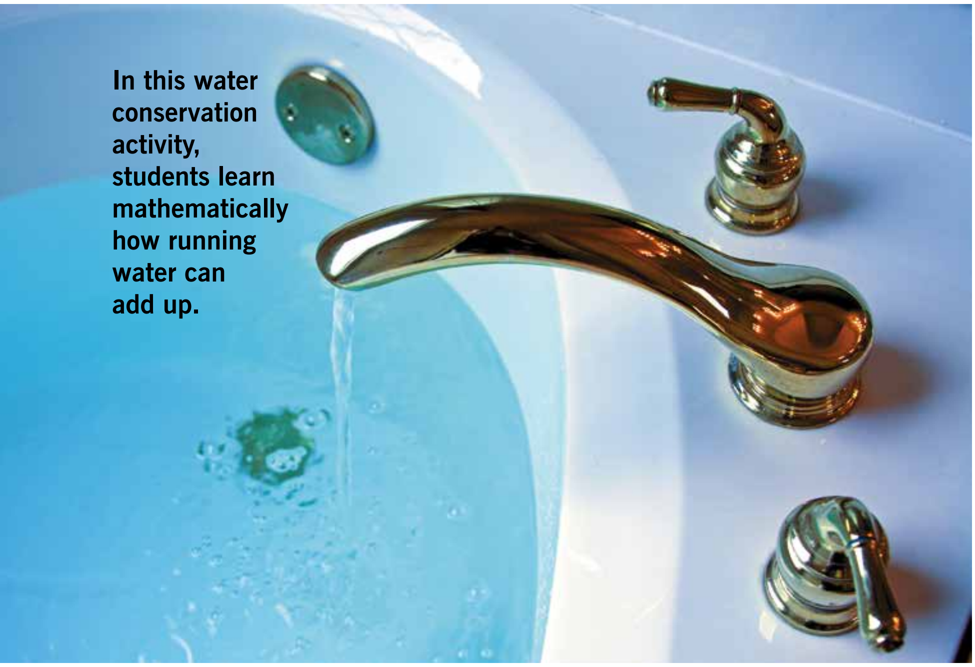


As discussed above, one change we would make is to be more explicit about the reason for engaging in the task. For instance, if the goal is for an individual to decide whether to bathe or shower, then the arithmetic model is adequate, but if the goal is to develop a general method to apply across contexts or to create a public awareness campaign, then more sophisticated models may be needed. Thus, when working with students in the classroom, their models should be shared and the advantages and disadvantages of each approach relative to the broader purpose should be discussed, including possible modifications to the models to serve other purposes. An explicit part of this discussion should be the reasonableness of the assumptions made in the students' models. Teachers can support students by asking:

- Do you agree with the length of the shower or capacity of a tub assumed in this model? Would someone with a different tub and shower be able to use this strategy? Explain your reasoning.
- Is it reasonable to conclude that everyone should take showers instead of baths?
- Explain how this strategy would or would not help an environmental group develop a clear public awareness message.

One potential reformulation of the model that is more general than the strategies shared previously is shown in **figure 4**. This model includes the best-case and worst-case flow rates for showerheads (the *S* functions) and the minimum and maximum bathtub capacities (the *B* functions) identified by the students. The intersection points between the *S* and *B* functions show break-even shower length times under various scenarios.

There are other potential obstacles



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for students and teachers when they initially engage in modeling. First, a key element of modeling is making reasonable and relevant assumptions, which is subjective. Students may struggle with which assumptions are relevant. In addition, making assumptions involves introducing new information into the problem, which may conflict with students' experiences with traditional word problems (where all necessary information is provided). Second, students may pursue approaches that the teacher had not anticipated, thus demanding a high level of flexibility from the teacher. Third, modeling activities are open-ended and can usually be improved further, so that students must be encouraged to develop the habit of revisiting their models and considering how they can be improved with new information, approaches, and insights. We emphasize that the modeling process is not linear and that it requires revisiting the elements of the cycle shown in **figure 1**.

STUDENTS' POSTUNIT VIEWS OF MODELING

On the postquestionnaire, the students focused less on modeling as multiple representations and continued to emphasize modeling as real-world problems. In addition, there were some indications that the students had begun to consider more complex real-world contexts for modeling, in particular, the idea that modeling is a process that involves making assumptions. When asked to define it, one student wrote that "to model with mathematics means to apply math to a real-world situation, and to have to use generalities and assumptions instead of a given equation"; another stated that "if you have a problem, without much information given, you can model the problem using assumptions and solve using math based on these assumptions."

RECOMMENDATIONS: INTEGRATE MODELING THROUGHOUT MATHEMATICS

Our experience showed that students with little to no experience with modeling may misconceive of it as simply the use of multiple representations or manipulatives. However, we also found that presenting a relatively short unit focused on the modeling cycle, using open-ended problems, and emphasizing the role of assumptions in modeling were able to help students begin to develop a deeper understanding of the modeling process. The idea that modeling involves making assumptions about real-world contexts is key. Rather than treating modeling as a stand-alone unit, we recommend integrating tasks throughout teacher preparation courses, in professional development experiences, and when teaching K–grade 12 students for several reasons.

First, modeling is intended to be integrated across Common Core content areas (Koestler et al. 2013; CCSSI 2010). Second, integrating modeling throughout the course allows revisiting some tasks and approaching them with different mathematical tools (e.g., algebra and geometry). Third, there is the need for students to regularly engage in real-world modeling activities throughout mathematics because there is ample evidence that they generally tend to ignore real-world considerations in mathematics class (Verschaffel, Greer, and De Corte 2000). Because the teacher is of particular importance in helping students understand the context, questioning their assumptions, and considering whether a model is adequate or should be revised, it is critical that we prepare teachers to feel confident in engaging in and teaching the modeling process.

CCSSM Practices in Action

- Understand rates and ratios and use them to solve real-world problems (6.RP.2, 6.RP.3, 7.RP.1, 7.RP.2)
- Solve real-life problems using numerical and algebraic expressions and equations (7.EE.3, 7.EE.4)

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Any thoughts on this article? Send an email to ***mtms@nctm.org*** and let us know.—*Ed.*



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