MOTION OF A SURFACE IN FOUR DIMENSIONAL SPACE

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STRUCTURE AND EVOLUTION EQUATIONS :

$$\Sigma \stackrel{\text{def}}{=} \{\mathbf{x} \in R^4 \ : \ x^j(u^\alpha)\} \quad \text{where} \quad j = 1,2,3,4 \quad \text{and} \quad \alpha = 1,2 \quad . \tag{1}$$

Tangent vectors,

$$\mathbf{t}_\alpha \stackrel{\text{def}}{=} \frac{\partial \mathbf{x}}{\partial u^\alpha} \qquad ; \qquad \alpha = 1,2 \ . \tag{2}$$

The metric, and surface area

$$g_{\alpha\beta} \stackrel{\text{def}}{=} < \mathbf{t}_\alpha, \mathbf{t}_\beta > \qquad ; \qquad g \stackrel{\text{def}}{=} \sqrt{det\,(g_{\alpha\beta})} \quad . \ . \tag{3}$$

The normal vector to the surface

$$\mathbf{n} \stackrel{\text{def}}{=} \frac{\Delta_g \mathbf{x}}{|\Delta_g \mathbf{x}|} \ . \tag{4}$$

Binormal vector? Form a frame on the surface $[\mathbf{t}_1 \ , \ \mathbf{t}_2 \ , \ \mathbf{n} \ , \ \mathbf{b}]$

COMPLEX VERSION : A complex vector bundle over the surface,

$$\mathbf{m} \stackrel{\text{def}}{=} \mathbf{n} + i\mathbf{b} \qquad ; \qquad \lambda_{\alpha\beta} \stackrel{\text{def}}{=} \kappa_{\alpha\beta} + i\tau_{\alpha\beta} \ , \tag{5}$$

and ... structure equations

$$\partial_\alpha \mathbf{t}_\beta = \Gamma_{\alpha\beta}{}^{\gamma} \mathbf{t}_\gamma + \frac{1}{2}\left[\lambda_{\alpha\beta}\overline{\mathbf{m}} + \overline{\lambda}_{\alpha\beta}\mathbf{m}\right] \quad : \quad \partial_\alpha \mathbf{m} = -\lambda_\alpha{}^{\gamma}\mathbf{t}_\gamma + iA_\alpha \mathbf{m} \ . \tag{6}$$

GAUGE INVARIANCE : $\theta(u^\alpha)$, defined on the surface Σ, rotate the vector $\mathbf{m}$ and the complex tensor $\lambda_{\alpha\beta}$ by $e^{i\theta}$, the new quantities will be denoted by the same name

$$\lambda_{\alpha\beta} \stackrel{\text{def}}{=} e^{i\theta}\left(\kappa_{\alpha\beta} + i\tau_{\alpha\beta}\right) \qquad ; \qquad \mathbf{m} \stackrel{\text{def}}{=} e^{i\theta}\left(\mathbf{n} + i\mathbf{b}\right) \ . \tag{7}$$

Complex scalar mean curvature Ψ and a gauge field A_α on the surface

$$A_\alpha \stackrel{\text{def}}{=} A_\alpha - \partial_\alpha\theta \qquad ; \qquad \Psi \stackrel{\text{def}}{=} g^{\alpha\beta}\lambda_{\alpha\beta} \ . \tag{8}$$

NOTATION (covariant derivatives)

$$\partial^A_\alpha \stackrel{\text{def}}{=} \partial_\alpha - iA_\alpha \quad ; \quad \alpha = 1, 2 \ , \qquad ()$$

STRUCTURE EQUATIONS :

$$\partial_\alpha \mathbf{t}_\beta = \Gamma_{\alpha\beta}{}^{\gamma} \mathbf{t}_\gamma + \frac{1}{2}\left[\lambda_{\alpha\beta}\overline{\mathbf{m}} + \overline{\lambda}_{\alpha\beta}\mathbf{m}\right] \quad ; \quad \partial^A_\alpha \mathbf{m} = -\lambda_\alpha{}^{\gamma}\mathbf{t}_\gamma \ . \tag{9}$$

Form the traceless complex tensor (Hopf)

$$\mu_{\alpha\beta} \stackrel{\text{def}}{=} \lambda_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}\Psi \ . \tag{10}$$

The Gauss, Codazzi-Mainardi and torsion equations respectively are

$$R = \langle \mu \ ; \overline{\mu} \rangle - \frac{1}{2}\left|\Psi\right|^2 \quad ; \quad \nabla^A_\beta \mu^\beta{}_\alpha = \frac{1}{2}\nabla^A_\alpha \Psi \quad ; \quad \nabla_{[\alpha}A_{\beta]} = [\mu \ ; \overline{\mu}]_{\alpha\beta} \ . \tag{11}$$

The torsion on the surface is

$$T := \epsilon^{\alpha\beta}[\mu \ ; \overline{\mu}]_{\alpha\beta} \ .$$

In general not pure gauge.

CONFORMAL COORDINATES :

$$ds^2 = g(z,\overline{z})dzd\overline{z} \ , \tag{12}$$

where $g(z,\overline{z})$ is the conformal factor. Complex differentiation

$$\partial_{\overline{z}} \stackrel{\text{def}}{=} \frac{1}{2}\big(\partial_1 + i\partial_2\big) \quad ; \quad \partial_z \stackrel{\text{def}}{=} \frac{1}{2}\big(\partial_1 - i\partial_2\big) \ , \tag{13}$$

and define a real potential σ from g and complex functions from A^α, $\mu_{\alpha\beta}$ as follows,

$$\sqrt{g} \stackrel{\text{def}}{=} e^\sigma \quad ; \quad a \stackrel{\text{def}}{=} \frac{1}{2}\big(A_1 - iA_2\big) \quad ; \quad \mu \stackrel{\text{def}}{=} \mu_{11} + i\mu_{12} \quad ; \quad \mu^* \stackrel{\text{def}}{=} \mu_{11} - i\mu_{12} \ . \tag{14}$$

The Conformal version of Gauss, Codazzi-Mainardi and torsion equations

$$\Delta\sigma = \frac{1}{2}\big(|\mu|^2 + |\mu^*|^2\big)e^{-2\sigma} - \frac{1}{4}\big|\Psi\big|^2 e^{2\sigma} \tag{15,a}$$

$$\big(\partial_z - ia\big)\mu = \frac{1}{2}g\big(\partial_{\overline{z}} - i\overline{a}\big)\Psi \quad ; \quad \big(\partial_{\overline{z}} - i\overline{a}\big)\mu^* = \frac{1}{2}g\big(\partial_z - ia\big)\Psi \tag{15,b}$$

$$i\big(\partial_{\overline{z}}a - \partial_z\overline{a}\big) = -\frac{1}{4g}\left[|\mu|^2 - |\mu^*|^2\right] \ . \tag{15,c}$$

The torsion is given by

$$T := -\frac{1}{4g}\left[|\mu|^2 - |\mu^*|^2\right] \ .$$

MOTION BY MEAN CURVATURE :

$$\partial_t \mathbf{x} = \frac{1}{2}\left[\Psi \overline{\mathbf{m}}_c + \overline{\Psi} \mathbf{m}_c\right] + X^\gamma \mathbf{t}_\gamma \quad ; \quad c = e^{ia} \ . \tag{16}$$

Binormal means $a = \pi/2$. Evolution of metric,

$$\partial_t g_{\alpha\beta} = 2N_\alpha{}^\gamma g_{\gamma\beta} + \mathcal{L}_X g_{\alpha\beta}$$

($c = i$ Binormal direction). How to choose X?
Evolution of mean curvature,

$$i\partial_t \Psi - i\mathcal{L}_X^A \Psi = \Delta_g^A \Psi + \frac{1}{2}\left[\langle \mu ; \overline{\mu} \rangle \Psi - \langle \mu \ ; \mu \rangle \overline{\Psi}\right] - B\Psi \ . \tag{17}$$

Evolution of Gauge fields,

$$\partial_t A_\alpha - \partial_\alpha B = \nabla_\beta P_\alpha{}^\beta + \left[\mu ; \overline{\mu}\right]_{\alpha\beta} X^\beta$$

$P \sim \Psi\mu$

Conservation laws, Gauss curvature R,

$$\partial_t R - X^\gamma \nabla_\gamma R + 2\nabla_\alpha \nabla_\beta \{Q^{\alpha\beta}\} = 0 \ . \tag{18}$$

The square mean curvature satisfies an evolution equation of the form

$$\partial_t\left(\left|\Psi\right|^2/2\right) - X^\gamma \nabla_\gamma\left(\left|\Psi\right|^2/2\right) + \langle Q \ ; P \rangle = \nabla_\alpha J^\alpha \ , \tag{19}$$

where the current J_α is defined by

$$J_\alpha \stackrel{\text{def}}{=} \frac{1}{2i}\left(\overline{\Psi}\nabla_\alpha^A \Psi - \Psi \overline{\nabla}_\alpha^A \overline{\Psi}\right) \ . \tag{20}$$

and $< P, Q > \sim (\Psi\mu)^2$.

CONFORMAL AGAIN : This requires an appropriate choice of the Lie vector field X.

$$ds^2 = g(t, z, \overline{z}) dz d\overline{z} \ , \tag{21}$$

We can form, from the vector field X^α and the matrices $P_{\alpha\beta}$ and $Q_{\alpha\beta}$, the complex functions

$$X \stackrel{\text{def}}{=} X^1 + iX^2 \quad ; \quad P \stackrel{\text{def}}{=} P_{11} + iP_{12} \quad ; \quad Q \stackrel{\text{def}}{=} Q_{11} + iQ_{12} \ , \tag{22}$$

and a short calculation shows that,

$$P = \frac{1}{2}\left[\mu\overline{\Psi} + \overline{\mu^*}\Psi\right] \qquad ; \qquad Q = \frac{1}{2i}\left[\mu\overline{\Psi} - \overline{\mu^*}\Psi\right] \ . \tag{23}$$

The complex function X has to satisfy the first order equation

$$\partial_{\overline{z}} X = -Q/g \ . \tag{24}$$

The evolution of g, $a = (1/2)(A_1 - iA_2)$, B and Ψ are given by the following set of equations,

$$\big\langle D_z, X \big\rangle \stackrel{\text{def}}{=} \partial_z X + \partial_{\overline{z}} \overline{X} \quad ; \quad \big\langle X, D_z \big\rangle \stackrel{\text{def}}{=} X\partial_z + \overline{X}\partial_{\overline{z}} \tag{25}$$

$$\partial_t g = \big\langle D_z, X \big\rangle \quad ; \quad \partial_t a - \partial_z B = \frac{1}{g}\left(\partial_{\overline{z}}\overline{P}\right) + \frac{i}{8g}\left(|\mu|^2 - |\mu^*|^2\right)\overline{X} \tag{26}$$

$$i\partial_t\big(g\Psi\big) - i\big\langle D_z, X\Psi \big\rangle - \big\langle X, a \big\rangle \Psi = \Delta^a \Psi + \frac{i}{g}\left[Q\mu^* + \overline{Q}\mu\right] - gB\Psi \tag{26}$$

where the operator Δ^a is defined below (covariant Laplacian)

$$\Delta^a \stackrel{\text{def}}{=} 2\big\langle D_z, D_{\overline{z}} \big\rangle - 4i\big\langle \overline{a}, D_z \big\rangle - |a|^2 - 2i\big\langle D_z, \overline{a}_{,z} \big\rangle$$

RADIAL SYMMETRY : An interesting special case arizes when we examine the equations of motion under the assumption of radial symmetry,

$$h(t,r)\big(dr^2 + r^2 d\phi^2\big) \ . \tag{2.1}$$

Notice that $g = rh$. Thus we can label the components of tensors using the indices r and ϕ. First notice that ∂_z, $\partial_{\overline{z}}$ become in radial coordinates,

$$2\partial_{\overline{z}} = e^{i\phi}\big(\partial_r + \frac{i}{r}\partial_\phi\big) \quad ; \quad 2\partial_z = e^{-i\phi}\big(\partial_r - \frac{i}{r}\partial_\phi\big) \ . \tag{2.2}$$

write μ_{rr}, $\mu_{r\phi}$ for the corresponding coordinate entries of the tensor $\mu_{\alpha\beta}$ and define

$$\pi \stackrel{\text{def}}{=} \mu_{rr} + \frac{i}{r}\mu_{r\phi} \quad ; \quad \pi^* \stackrel{\text{def}}{=} \mu_{rr} - \frac{i}{r}\mu_{r\phi} \ . \tag{2.3}$$

Express μ, μ^* in terms of π and π^*,

$$\mu = \pi e^{2i\phi} \qquad ; \qquad \mu^* = \pi^* e^{-2i\phi} \ . \tag{2.4}$$

Radial symmetry means

$$\Psi(t,r) \quad ; \quad \mu_{r\phi} = 0 \quad ; \quad A_\phi = 0$$

the torsion is zero. The Coulomb gauge implies that $A_r = 0$ and the Codazzi-Mainardi equations become

$$\partial_r\big(r^2\pi\big) = \frac{1}{2}hr^2\partial_r\Psi \ . \tag{2.5}$$

Integrating the equation above we obtain

$$\pi(t,r) = \frac{1}{2}h\Psi - \frac{1}{2r^2}\int_0^r (s^2h)_{,s}\big(\Psi(t,s) - \Psi(t,0)\big)\ ds \ . \tag{2.6}$$

define a complex function Φ as follows

$$\Phi(t,r) \stackrel{\text{def}}{=} \frac{1}{r^2 h} \int_0^r (s^2 h)_{,s} \big[\Psi(t,s) - \Psi(t,0)\big] \; ds \tag{2.7}$$

Now define two real potentials V and W, using Φ and Ψ,

$$V = \frac{1}{4}\big(\Psi\overline{\Phi} + \overline{\Psi}\Phi\big) \quad ; \quad W = \frac{1}{4i}\big(\Psi\overline{\Phi} - \overline{\Psi}\Phi\big) \; . \tag{2.8}$$

The vector field $X^1 + iX^2$ must satisfy, in polar coordinates, $X^\phi = 0$, call $X^r = Y(t,r)$ and $X = Ye^{i\phi}$

$$\partial_r\big(Y/r\big) = -W/r \qquad ; \qquad \partial_t g = \frac{1}{r}\partial_r\big(rY\big) \tag{2.10}$$

therefore $h(t,r)$ satisfies the integrodifferential equation

$$\partial_t h = -W - 2\int_0^r \big(W(t,s)/s\big) \; ds \; . \tag{2.11}$$

The potential B can be computed

$$-\partial_r B = \frac{1}{hr^2}\partial_r \left(hr^2\big(\frac{1}{2}\big|\Psi\big|^2 - V\big)\right) \tag{2.12}$$

Evolution of $\Psi(t,r)$

$$i\partial_t\Psi + i\left(r\int_0^r (W/s)ds\right)\partial_r\Psi = \frac{1}{h}\Delta\Psi + \big(-B + iW\big)\Psi - iW\Phi \; , \tag{2.14}$$

where Δ is the radial Laplacian.

SPECIAL SOLUTIONS : Torus

$$\mathbf{r}_\alpha(t,u^\alpha) = R_\alpha(t)\exp[iu^\alpha/R_\alpha(t)] \qquad \alpha = 1,2 \tag{3.1}$$

where we made the identification $\mathbf{r}_1 = x^1 + ix^2$ and $\mathbf{r}_2 = x^3 + ix^4$. The time derivative of the surface is

$$\partial_t \mathbf{r}_\alpha = R'_\alpha \exp[iu^\alpha/R_\alpha] - i\frac{u^\alpha R'_\alpha}{R_\alpha}\exp[iu^\alpha/R_\alpha] \ . \tag{3.2}$$

Using the binormal vector the evolution of the surface must be described by

$$\partial_t \mathbf{r}_1 = H\sin\theta\ \mathbf{n}_1 + X^1\mathbf{s}_1 \qquad ; \qquad \partial_t \mathbf{r}_2 = -H\sin\theta\ \mathbf{n}_2 + X^2\mathbf{s}_2 \ . \tag{3.3}$$

Notice that $H_\alpha = 1/R_\alpha$ and mean curvature is $H = R_1R_2/\sqrt{R_1^2+R_2^2}$. The evolution equations for R_1 and R_2

$$R'_1 = \frac{-1}{R_1R_2}R_1 \quad ; \quad R'_2 = \frac{1}{R_1R_2}R_2 \ . \tag{3.4}$$

Notice that $R_1R_2 = C$ i.e. the product of the two radii is constant, therefore set $p = 1/R_1R_2$, which is the inverse of the area,

$$R_1(t) = R_1(0)e^{-pt} \qquad ; \qquad R_2(t) = R_2(0)e^{pt} \tag{3.5}$$

The initial angle is $\cos\theta_0 = H_1(0)/\sqrt{H_1^2(0)+H_2^2(0)}$ mean curvature is

$$\Psi(t) = H(0)q(t)e^{i\theta(t)} \tag{3.6}$$

where $q(t)$ and $\theta(t)$ are

$$q(t) = \sqrt{\cosh(2pt) + \cos(2\theta_0)\sinh(2pt)}\ ;\ \theta(t) = \arctan\left(\tan(\theta_0)e^{-2pt}\right) \ . \tag{3.7}$$

Notice that the mean curvature blows up but not in finite time These solutions are elementary, to produce more interesting solutions we have to address the problem of representing a surface embedded in R^4, the idea is that elementary solutions of the evolution equations might give rise to interesting evolving surfaces.

QUATERNIONS AND MOVING FRAMES : The quaternions can be realized as an Algebra of 2×2 complex matrices that are linear combinations of the following basis,

$$\mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad , \quad \mathbf{i} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \quad , \quad \mathbf{j} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad , \quad \mathbf{k} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} . \tag{C.1}$$

for $(p_0, p_1, p_2, p_4) \in R^4$ we can write $\mathbf{p} = p_0\mathbf{1} + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$ i.e. we form the 2×2 complex matrix,

$$\mathbf{p} = \begin{bmatrix} p_0 + ip_1 & p_2 + ip_3 \\ -p_2 + ip_3 & p_0 - ip_1 \end{bmatrix} . \tag{C.2}$$

The algebra of quaternions is noncommutative, the algebraic relations between the elements of the basis are,

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1} \qquad ; \qquad \mathbf{ij} = \mathbf{k} \ ; \ \mathbf{jk} = \mathbf{i} \ ; \ \mathbf{ki} = \mathbf{j} \ , \tag{C.4}$$

moreover $\mathbf{ij} + \mathbf{ji} = \mathbf{jk} + \mathbf{kj} = \mathbf{ki} + \mathbf{ik} = \mathbf{0}$

For a given surface Σ in R^4, let us denote by $\mathbf{x}(u^1, u^2)$ the position vector as a function of local coordinates and make the identification

$$\mathbf{x} = x^1\mathbf{1} + x^2\mathbf{i} + x^3\mathbf{j} + x^4\mathbf{k} . \tag{C.6}$$

FRAME at each point on the surface, $[\mathbf{x}_{,1}; \mathbf{x}_{,2}; \mathbf{n}; \mathbf{b}]$ where $[\mathbf{x}_{,1}; \mathbf{x}_{,2}]$ are tangent while $[\mathbf{n}; \mathbf{b}]$ are the normal and binormal vectors.

CONFORMAL METRIC on the surface,

$$< \mathbf{t}_1, \mathbf{t}_1 >=< \mathbf{t}_2, \mathbf{t}_2 >= g \quad \text{and} \quad < \mathbf{t}_1, \mathbf{t}_2 >= 0 \ . \qquad (C.7)$$

MAP from the frame $\{\mathbf{b}, \mathbf{n}, \mathbf{t}_1, \mathbf{t}_{,2}\}$ to the basis $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$

$$\mathbf{b} \mapsto \mathbf{1} \quad ; \quad \mathbf{n} \mapsto \mathbf{i} \quad ; \quad \frac{\mathbf{x}_{,2}}{\sqrt{g}} \mapsto \mathbf{j} \quad ; \quad \frac{\mathbf{x}_{,1}}{\sqrt{g}} \mapsto \mathbf{k} \ , \qquad (C.8)$$

Rotation in R^4 (Hamilton) is right-left multiplication

$$\big[\mathbf{b} \ ; \ \mathbf{n} \ ; \ \mathbf{x}_{,2}/\sqrt{g} \ ; \ \mathbf{x}_{,1}/\sqrt{g}\big] = L(\mathbf{p})\big[\mathbf{1} \ ; \ \mathbf{i} \ ; \ \mathbf{j} \ ; \ \mathbf{k}\big]R(\mathbf{q}) \ . \qquad (C.9)$$

Unit Quaternions,

$$L(\mathbf{p})\mathbf{x} = \mathbf{p}\mathbf{x} \quad ; \quad R(\mathbf{q})\mathbf{x} = \mathbf{x}\mathbf{q} \quad ; \quad |\mathbf{p}| = 1 \quad ; \quad |\mathbf{q}| = 1 \ . \qquad (C.10)$$

$$\mathbf{x}_{,\overline{z}} = \sqrt{g}L \begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix} R \quad ; \quad \mathbf{x}_{,z} = \sqrt{g}L \begin{bmatrix} 0 & 0 \\ i & 0 \end{bmatrix} R \qquad (C.12, a)$$

$$\mathbf{m} = L \begin{bmatrix} 2i & 0 \\ 0 & 0 \end{bmatrix} R \quad ; \quad \overline{\mathbf{m}} = L \begin{bmatrix} 0 & 0 \\ 0 & -2i \end{bmatrix} R \ . \qquad (C.12, b)$$

The structure equations on the surface (recall the quantities)

$$\sqrt{g} = e^{\sigma} \quad ; \quad \mu = \mu_{11} + i\mu_{12} \quad ; \quad \mu^* = \mu_{11} - i\mu_{12} \ . \qquad (C.13)$$

$$\mathbf{x}_{,\overline{z}z} = \frac{1}{8} g \left(\Psi\overline{\mathbf{m}} + \overline{\Psi}\mathbf{m}\right) \qquad ()$$

$$\mathbf{x}_{,zz} = 2\sigma_{,z}\mathbf{x}_{,z} + \frac{1}{4}\left(\mu^*\overline{\mathbf{m}} + \overline{\mu}\mathbf{m}\right)$$

$$\mathbf{x}_{,\overline{z}\overline{z}} = 2\sigma_{,\overline{z}}\mathbf{x}_{,\overline{z}} + \frac{1}{4}\left(\mu\overline{\mathbf{m}} + \overline{\mu^*}\mathbf{m}\right) \qquad ()$$

$$\mathbf{m}_{,\overline{z}} = -\frac{1}{2}\Psi \ \mathbf{x}_{,\overline{z}} - \frac{\mu}{g} \ \mathbf{x}_{,z} + i\overline{a}\mathbf{m}$$

$$\mathbf{m}_{,z} = -\frac{1}{2}\Psi \ \mathbf{x}_{,z} - \frac{\mu^*}{g} \ \mathbf{x}_{,\overline{z}} + ia\mathbf{m} \ , \qquad ()$$

a is a complex gauge vector field i.e. $a = (A_1 - iA_2)/2$.

EVOLUTION OF FRAME : Only Ψ, g and a,

$$\mathbf{p}_l := \begin{bmatrix} \psi_l & -\overline{\phi_l} \\ \phi_l & \overline{\psi_l} \end{bmatrix} \quad ; \quad \mathbf{p}_r := \begin{bmatrix} \psi_r & -\overline{\phi_r} \\ \phi_r & \overline{\psi_r} \end{bmatrix}$$

Evolution equation (not in time)

$$\big(\partial_z - ia/2\big)\psi_l = -\widehat{\Psi}\phi_l \quad ; \quad \big(\partial_{\overline{z}} - i\overline{a}/2\big)\psi_r = -\widehat{\Psi}\phi_r \qquad (C.39, a)$$

$$\big(\partial_{\overline{z}} + i\overline{a}/2\big)\phi_l = \overline{\widehat{\Psi}}\psi_l \quad ; \quad \big(\partial_z + i\overline{a}/2\big)\phi_r = \overline{\widehat{\Psi}}\psi_r \qquad (C.39, b)$$

where $\widehat{\Psi} = \sqrt{g}\Psi/4$.

GAUGE INVARIANCE

$$\psi^l \mapsto \psi^l e^{i\theta/2} \quad ; \quad \phi^l \mapsto \phi^l e^{-i\theta/2} \qquad (C.40, a)$$

$$\psi^r \mapsto \psi^r e^{-i\theta/2} \quad ; \quad \phi^r \mapsto \phi^r e^{i\theta/2} \qquad (C.40, b)$$

$$a \mapsto a + \theta_{,z} \quad ; \quad \widehat{\Psi} \mapsto \widehat{\Psi} e^{i\theta} \ . \qquad (C.40, c)$$

COULOMB gauge means $\partial_1 A_1 + \partial_2 A_2 = 0$ and this translates to $a = i\omega_{,z}$. Renormalizing

$$\psi_l = \psi_l e^{\omega/2} \qquad ; \qquad \phi_l = \phi_l e^{\omega/2} \qquad (C.41, a)$$

$$\psi_r = \psi_r e^{-\omega/2} \qquad ; \qquad \phi_r = \phi_r e^{-\omega/2} \qquad (C.41, b)$$

we have the system of equations,

$$\partial_z \psi_l = -\widehat{\Psi}\phi_l \quad ; \quad \partial_{\overline{z}}\psi_r = -\widehat{\Psi}\phi_r \qquad (C.42, a)$$

$$\partial_{\overline{z}}\phi_l = \overline{\widehat{\Psi}}\psi_l \quad ; \quad \partial_z \phi_r = \overline{\widehat{\Psi}}\psi_r \ . \qquad (C.42, b)$$

TORSION is given by,

$$2\omega = \log\left(\frac{|\psi_l|^2 + |\phi_l|^2}{|\psi_r|^2 + |\phi_r|^2}\right) \ . \qquad (C.43)$$

Take the Laplacian of ω.

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CONSTANT CURVATURE : We want to solve,

$$\partial_z \psi_l = -\widehat{\Psi}\phi_l \quad ; \quad \partial_{\overline{z}} \psi_r = -\widehat{\Psi}\phi_r \quad ; \quad \widehat{\Psi} = \frac{1}{4}\sqrt{g}\Psi \qquad ()$$

$$\partial_{\overline{z}} \phi_l = \overline{\widehat{\Psi}}\psi_l \quad ; \quad \partial_z \phi_r = \overline{\widehat{\Psi}}\psi_r \ . \qquad ()$$

$\omega = -2ns$ (no torsion), write $u^1 = s$, $u^2 = u$ the mean curvature,

$$\Psi = H \exp[2i(\theta_0 + nu)] \qquad ()$$

where H, n and θ_0 are constants. Assume the quaternions have the following form

$$\psi_l = p(s) \exp[i(n-m)u + i\theta_0 - ns] \qquad ()$$

$$\phi_l = q(s) \exp[-i(n+m)u - i\theta_0 - ns] \qquad ()$$

$$\psi_r = p(s) \exp[i(n+m)u + i\theta_0 + ns] \qquad ()$$

$$\phi_r = q(s) \exp[-i(n-m)u - i\theta_0 + ns] \ , \qquad ()$$

system of ordinary differential equations for $p(s)$ and $q(s)$

$$p' - mp = -\frac{1}{2}H\left(|p|^2 + |q|^2\right)q \qquad ()$$

$$q' + mp = \frac{1}{2}H\left(|p|^2 + |q|^2\right)p \ . \qquad ()$$

The system above is Hamiltonian. Conserved quantities,

$$E := \frac{1}{4}H\left(|p|^2 + |q|^2\right)^2 - m\left(p\overline{q} + \overline{p}q\right) \qquad ()$$

$$M = \frac{1}{i}\left(p\overline{q} - \overline{p}q\right) \ . \qquad ()$$

Assume that $M = 0$ to simplify matters,

$$p(s) = p(s)e^{i\alpha} \qquad ; \qquad q(s) = q(s)e^{i\alpha} \tag{}$$

for some fixed angle α and $p(s)$, $q(s)$ real functions.

$$E = \frac{1}{4}H\big(p^2 + q^2\big)^2 - 2m\big(pq\big) \tag{}$$

We have **three critical points :**

$$(0, 0) \qquad ; \qquad \big(\pm\sqrt{m/H}, \pm\sqrt{m/H}\big)\ . \tag{}$$

Evaluating E at the critical points

$$E(0, 0) = 0 \qquad ; \qquad E\big(\pm\sqrt{m/H}, \pm\sqrt{m/H}\big) = -m/H$$

thus $(\pm\sqrt{m/H}, \pm\sqrt{m/H})$ are minima, while $(0, 0)$ is a saddle point, hence there is a homoclinic orbit emanating and ending at $(0, 0)$ while near

$$(\pm\sqrt{m/H}, \pm\sqrt{m/H})$$

there are periodic solutions. Level curves of $E(p, q)$ for $-m/H < E < 0$. Let us call $(p(s), q(s))$ such a periodic solution.
These give rise to solutions

$$(x^1 + ix^2)_{,\bar{z}} = i\psi_l\phi_r \qquad ; \qquad (x^3 + ix^4)_{,\bar{z}} = i\psi_l\overline{\psi_r} \tag{}$$

$$(x^1 + ix^2)_{,z} = i\phi_l\psi_r \qquad ; \qquad (x^3 + ix^4)_{,z} = -i\phi_l\overline{\phi_r}\ . \tag{}$$