Spaces of commuting elements in Lie groups

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General:

- π = fin. gen. discrete group.
- G = Lie group.
- $\mathrm{Hom}(\pi, G)$ = space of homomorphisms (gp.) $\pi \longrightarrow G$
- $\mathrm{Rep}(\pi, G)$ = $\mathrm{Hom}(\pi, G)/G$ = representation space of π into G.
- The projection $F_n \longrightarrow \pi$ induces an inclusion $\mathrm{Hom}(\pi, G) \hookrightarrow G^n$. Hence we endow $\mathrm{Hom}(\pi, G)$ with the subspace topology in G^n.
- $\mathrm{Hom}(\pi, G)$ is the fixed point set of the action of π, on the pointed mapping space $\mathrm{Map}_*(\pi, G)$, defined by

$$g \cdot f(h) = f(hg)\, f(g)^{-1}.$$

* If π = finite then $\mathrm{Hom}(\pi, G)$ is the fixed pt. set of a smooth action of π on a cpct. mfld., $G^{|\pi|-1}$, hence is itself a smooth mfld.

Examples:

- $\mathrm{Hom}(\mathbb{Z}, G) = G$
- $\mathrm{Hom}(\mathbb{Z}^n, G) = G^n$ if G is abelian
- $\mathrm{Hom}(\mathbb{Z}^n, SO(3)) = \mathrm{Hom}(\mathbb{Z}^n, SO(3))_1 \sqcup \left(\coprod_{c(n)} S^3/Q_8 \right)$ — $c(n)$ increases exponentially with n.
  (Sjerve, Torres-Giese)
- S = closed surface. The Teichmüller space $T(S)$ can be realized as a connected component in the representation variety $\mathrm{Hom}(\pi_1(S), PSL(2,\mathbb{R}))/PSL(2,\mathbb{R})$ consisting of discrete embeddings.

- (Cohen-Adem-Gomez): G s.t. $\mathrm{Rep}(\mathbb{Z}^n, G)$ conn. for $n \geq 1$; Then

$\mathrm{Rep}(\mathbb{Z}^n, G) \cong T^n/W$

- $\rightarrow \mathrm{Rep}(\mathbb{Z}^n, U(m)) \simeq Sp^m((S^1)^n)$ $\quad (W \cong \Sigma_m)$
- $\rightarrow \mathrm{Rep}(\mathbb{Z}^n, Sp(m)) \cong Sp^m((S^1)^n/\mathbb{Z}/2)$.
- $\rightarrow \mathrm{Rep}(\mathbb{Z}^2, SU(m)) \cong \mathbb{C}P^{m-1}$
- $\rightarrow \mathrm{Rep}(\mathbb{Z}^2, Sp(m)) \cong \mathbb{C}P^m$.

Connectedness: Hom (π, G) is not always connected.

- (Goldman) $G = n$-fold cover of $PSL(2,\mathbb{R})$

  $\pi = \pi_1(S)$, S = surface (closed-oriented) of genus g.

  Then Hom (π, G) is not connected. $\begin{cases} \#: 2n^{2g} + (4g-4)/n - 1 & \text{if } n \mid 2g-2 \\ \#: 2\left[\frac{(2g-2)}{n}\right] + 1 & \text{if } n \nmid 2g-2. \end{cases}$

- (Rojo) # components of Hom$(\mathbb{Z}^k, O(n))$ and Hom$(\mathbb{Z}^k, SO(n))$.

  ↳ stabilizes as $n \gg k$, k = fixed.

  ↳ increases exponentially as k increases.

- (Adem-Cohen) If every abelian subgp. of G is contained in a path-connected abelian subgroup, then Hom $(\mathbb{Z}^n, G)$ is conn.

  eg: $G = U(n), SU(n), Sp(n)$. , neg.: $G = SO(3)$ contains $\mathbb{Z}/2 \oplus \mathbb{Z}/2$.

- In general: The map $B: \text{Hom}(\pi, G) \longrightarrow \text{Map}_*(B\pi, BG)$ sending $f \longmapsto Bf$ is cont. and induces a map of sets $B_0: \pi_0(\text{Hom}(\pi, G)) \longrightarrow [B\pi, BG]$, which factors through $\pi_0(\text{Rep}(\pi, G))$. ($\pi_0(\text{Hom}(\pi, G)) \longrightarrow \pi_0(\text{Rep}(\pi, G))$ is a bij. if G is conn.

- Problem: Give conditions on π & G s.t. $B_0: \pi_0(\text{Hom}(\pi, G)) \longrightarrow [B\pi, BG]$ is a bijection.

Homology / Cohomology: X = topological space, $H^n(X,R)$ = coh. in deg n, coeff in R

$H_n(X,R)$ = hom. " " , " ".

- $\tilde{H}_{n+1}(\Sigma X; R) \cong \tilde{H}_n(X; R)$, $n \geq 1$.
- $\tilde{H}_n(X \vee Y; R) \cong \tilde{H}_n(X) \oplus \tilde{H}_n(Y)$, $n \geq 1$.

Remark: In certain cases topological spaces split after suspending a number of times.

e.g.
- $\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y)$
- $\Sigma(\underbrace{X \times \cdots \times X}_{n}) \simeq \Sigma\left(\bigvee_{k=1}^{n} \binom{n}{k} X^{\wedge k}\right)$.

Idea: Decompose Hom (π, G) by suspending it and try to extract info about cohomology.

⊛ From now on assume $\pi = \mathbb{Z}^n$, unless otherwise stated.

Thm (Baird '07): G = compact & connected Lie group. Then

W = Weyl gp.
T = max. torus

- $H^*(\mathrm{Hom}(\mathbb{Z}^n, G)_1; \mathbb{Q}) \cong H^*(G/T \times T^n; \mathbb{Q})^W$ (the ring of invariants).
- Torsion divides the order of the Weyl group.

Remark: not much is known about the integral homology, in general; except for a few cases: $SU(2)$, $SO(3)$ in low degrees.

- (Pettet-Souto) G = reductive algebraic group, $K \subseteq G$ maximal compact subgp. Then $\mathrm{Hom}(\mathbb{Z}^n, K) \hookrightarrow \mathrm{Hom}(\mathbb{Z}^n, G)$ is a strong deformation retract.

  Bergeron generalized this to nilpotent groups Γ.

(*) From now on assume G = compact & connected.

Strategy:

- There is a map $\theta_n : G \times T^n \longrightarrow \mathrm{Hom}(\mathbb{Z}^n, G)$ defined by

$$(g, t_1, \dots, t_n) \longmapsto (g t_1 g^{-1}, \dots, g t_n g^{-1}).$$

θ_n is not a surjection, in general, but it is if we restrict to the image $\mathrm{Hom}(\mathbb{Z}^n, G)_1$.

$$\theta_n : G \times T^n \twoheadrightarrow \mathrm{Hom}(\mathbb{Z}^n, G)_1$$

$T \curvearrowright G \times T^n$ diagonally: $t \cdot (g, t_1, \dots, t_n) = (gt, t t_1 t^{-1}, \dots, t t_n t^{-1}) = (gt, t_1, \dots, t_n)$

Hence θ_n factors through $G/T \times T^n \Rightarrow \exists\, \bar{\theta}_n : G/T \times T^n \longrightarrow \mathrm{Hom}(\mathbb{Z}^n, G)_1$

Similarly $W \curvearrowright G/T \times T^n$ diagonally; Again $\bar{\theta}_n$ is W-invariant, hence factors through $G/T \times_W T^n$ to obtain a surjection

$$\hat{\theta}_n : G/T \times_W T^n \longrightarrow \mathrm{Hom}(\mathbb{Z}^n, G)_1.$$

Lemma: If $R = \mathbb{Z}[\frac{1}{|W|}]$, the homology of $\hat{\theta}_n^{-1}(pt)$ with coeff. in R is trivial.

Vietoris-Begle theorem $\Rightarrow$ $\hat{\theta}_{n*} : H_*(G/T \times_W T^n; R) \xrightarrow{\cong} H_*(\mathrm{Hom}(\mathbb{Z}^n, G)_1; R)$.

The James reduced product (construction):

X = topological space with basepoint.

Defn: James reduced product $= J(X) := \left(\coprod_{n\geq 0} X^n \right) \big/ \sim$, where $\sim$ is the equivalence relation ~~induced~~ generated by $(x_1, \ldots, x_n) \sim (x_1, \ldots, \hat{x}_i, \ldots, x_n)$ (i.e. i-th coord. removed) if $x_i = * =$ basepoint. (Also seen as a free monoid).

Fact: If X = path connected CW complex, then $\exists\, \theta : J(X) \to \Omega \Sigma X$, which is a homotopy equivalence.

Fact: Let $N = \tilde{H}_*(X; R)$, R = comm. ring w/ 1 s.t. reduced homology is R-free.

Bott-Samelson: $\mathcal{T}[N] \longrightarrow H_*(J(X); R)$

an isomorphism of algebras, where $\mathcal{T}(N)$ is the tensor algebra generated by N.

Apply this construction to our map $\hat{\theta}_n : G/T \times_W T^n \longrightarrow \mathrm{Hom}(\mathbb{Z}^n, G)_1$.

Define $\mathrm{Comm}(G)_1 := \left(\coprod_{n \geq 1} \mathrm{Hom}(\mathbb{Z}^n, G)_1 \right) \big/ \sim$, $\sim$ the same relation. $* = 1_G$.

(e.g. $\mathrm{Comm}(SO(3)) = \mathrm{Comm}(SO(3))_1 \amalg \left(\coprod_{\infty} S^3/Q_8 \right)$.)

Then we obtain a map:

$$\theta : G/T \times_W J(T) \longrightarrow\!\!\!\!\rightarrow \mathrm{Comm}(G)_1 \text{ (a surjection).}$$

- By previous methods: θ induces an iso. in homology with coeff. in $R = \mathbb{Z}\left[\frac{1}{|W|}\right]$.

  i.e. $$H_*(G/T \times_W J(T); R) \cong H_*(\mathrm{Com}(G)_1; R)$$

- Using a spectral sequence argument for: $(W \longrightarrow) G \times_T J(T) \to G \times_{NT} J(T) \to BW$
  $E^2_{s,t} = H_s(BW; H_t(G/T \times J(T); R))$.

  Collapses at the E^2-page.

  $E^2_{0,t} = H_0(BW; \ldots) = H_t(G/T \times J(T); R)_W$ (coinvariants.)

Theorem (Cohen–S.) $H_*(\mathrm{Comm}(G)_1; R) \xrightarrow{\cong} H_*(G/T \times J(T); R)_W$.

$$H^*(\mathrm{Comm}(G)_1; R) \xrightarrow{\cong} H^*(G/T \times J(T); R)^W.$$

(For simplicity one can pick $R = \mathbb{R}$ or $\mathbb{Q}$).

Recall: If $G = U(m), SU(m), Sp(m)$, then $\mathrm{Comm}(G)_1 = \mathrm{Comm}(G)$.

Hilbert-Poincaré series: Traditionally $T(X; \mathbb{R}) = \sum_{k \geq 0} (\mathrm{rank}_{\mathbb{R}} H^k(X; \mathbb{R}))\, t^k$.

Let $C^* = H^*(G/T; \mathbb{R})$

$\tilde{E} = \tilde{H}^*(T; \mathbb{R}) \cong \bigoplus_{k=1}^{n} \Lambda^k \mathbb{R}^n$

$\mathcal{T}^*[\tilde{E}] = \mathbb{R}$-dual of the tensor algebra gen. by $\tilde{E}$, the coh. of $J(T)$.

$W \curvearrowright C^* \otimes \mathcal{T}^*[\tilde{E}]$ diagonally (i.e. on cohomology of $G/T \times J(T)$).

Define a tri-graded Hilbert-Poincaré series:

$$\mathrm{Hilb}\left((C^* \otimes \mathcal{T}^*[\tilde{E}])^W; q, s, t\right) = \sum_{\substack{i, m \geq 0 \\ j = k_1 + \dots + k_m}} \dim_{\mathbb{R}}(M^W_{i,j,m})\, q^i s^j t^m$$

$\dim_{\mathbb{R}}(M^W_{i,j,m})$ is the rank of the invariant module
- coh. deg. i, in C^*
- tensordeg. m, in $\mathcal{T}^*[\tilde{E}]$
- hom. deg j, in $\mathcal{T}^*[\tilde{E}]$. (total)

i.e. $M_{i,j,m} = \bigoplus_{\substack{j = k_1 + \dots + k_m \\ k_q > 0}} (C_i \otimes \Lambda^{k_1}\mathbb{R}^n \otimes \dots \otimes \Lambda^{k_m}\mathbb{R}^n)$, $n > 0$.

Counting 3 things at the same time!

Theorem: $\mathrm{Hilb}\left((C^* \otimes \mathcal{T}^*[\tilde{E}])^W; q, s, t\right) = \mathrm{Hilb}(\mathrm{Comm}(G)_1; q, s, t) =$

$$= \frac{\prod_{i=1}^{n}(1 - q^{2d_i})}{|W|} \sum_{w \in W} \frac{1}{\det(1 - q^2 w)\,(1 - t(\det(1 + sw) - 1))}.$$

($d_1, \dots, d_n$ = characteristic / fundamental degrees of W)

Example: Hilbert-Poincaré series for $\mathrm{Comm}(G)$ when $G = U(2)$.

- $W \cong \Sigma_2 = \{1, w\}$
- $(d_1, d_2) = (1, 2)$
- $G/T \approx S^2$
- Sum runs over 1 and w.

$$\mathrm{Hilb}\left(\left(C^*\otimes J^*[\vec{E}]\right)^W; q,s,t\right) = \frac{(1-q^2)(1-q^4)}{2}\left(A_1 + A_w\right), \text{ where}$$

$$A_1 = \frac{1}{(1-q^2)^2\left(1-t\left((1+s)^2-1\right)\right)} \quad \text{and} \quad A_w = \frac{1}{(1-q^2)(1+q^2)\left(1-t\left[(1+s)(1-s)-1\right]\right)}.$$

$$\Rightarrow \mathrm{Hilb}(\ldots) = \frac{1+q^2}{2\left(1-t(s^2+2s)\right)} + \frac{1-q^2}{2\left(1+s^2t\right)}.$$

$\Rightarrow$ the coefficient of t^m, $m>0$, is

$$\frac{1}{2}\left[(1+q^2)(s^2+2s)^m + (1-q^2)(-s^2)^m\right] = \sum_{1\le j\le m} 2^{j-1}\binom{m}{j} s^{2m-j} + \begin{cases} s^{2m} & \text{if } m \text{ is even} \\ q^2 s^{2m} & \text{if } m \text{ is odd.} \end{cases}$$

Stable decompositions:

Theorem (Cohen-S.) $\Sigma(G\times_{NT} J(T)) \simeq \Sigma\left(G/NT \vee \left(\bigvee_{q\ge 1}\left(G\times_{NT}\hat{T}^q\right)/(G/NT)\right)\right)$

Theorem (Cohen-S.) $\Sigma(\mathrm{Comm}(G)) \simeq \bigvee_{n\ge 1}\Sigma(\widehat{\mathrm{Hom}}(\mathbb{Z}^n,G))$.

Theorem (Cohen-S.) $(G\times_{NT}\hat{T}^q)/_{G/NT} \xrightarrow{\simeq} \widehat{\mathrm{Hom}}(\mathbb{Z}^q,G)$ if $|W|$ is inverted.

Theorem (Adem-Cohen) $\Sigma\,\mathrm{Hom}(\mathbb{Z}^n,G) \simeq \Sigma\left(\bigvee_{1\le k\le n}\left[\bigvee_{\binom{n}{k}}\widehat{\mathrm{Hom}}(\mathbb{Z}^k,G)\right]\right)$.

- real coh. of $G\times_{NT}\hat{T}^m/(G/NT)$ is given by $\sum_{\substack{j=k_1+\cdots+k_m\\ i\ge 0}}\left(M_{i,j,m}\right)^W$

Corollary: Additively: $\tilde{H}^d(\mathrm{Hom}(\mathbb{Z}^m,G);\mathbb{R}) \cong \sum_{1\le s\le m}\sum_{i+j=d}\left(\sum_{\substack{j=k_1+\cdots+k_s\\ i\ge 0}}\bigoplus_{\binom{m}{s}}\left(M_{i,j,s}\right)^W\right)$.

Integral cohomology:

Let $H_{(i,j)} = \{(x_1,\ldots,x_n)\in G^n : [x_i,x_j]=1\} \cong \mathrm{Hom}(\mathbb{Z}^2,G)\times G^{n-2}$

Then:
- $\mathrm{Hom}(\mathbb{Z}^n,G) = \bigcap_{1\le i<j\le n} H_{(i,j)}$
- $G^n\setminus\mathrm{Hom}(\mathbb{Z}^n,G) = \bigcup_{1\le i<j\le n}\left(G^n\setminus\mathrm{Hom}(\mathbb{Z}^2,G)\times G^{n-2}\right)$

$\Rightarrow$ Apply Mayer-Vietoris spectral sequence.

(Baird-Jeffrey-Selick) compute $H^*(\mathrm{Hom}(\mathbb{Z}^n,SU(2));\mathbb{Z})$ using stable decompositions.