# THE MAYER-VIETORIS SPECTRAL SEQUENCE

MENTOR STAFA

ABSTRACT. In these expository notes we discuss the construction, definition and usage of the Mayer-Vietoris spectral sequence. We make these notes available hoping they are helpful to people looking for a definition or an example of this spectral sequence.

## CONTENTS



## 1. INTRODUCTION

The purpose of these notes is to outline a description of the Mayer-Vietoris spectral sequence, which is a spectral sequence constructed to compute the homology of a topological space $X$ given a cover $\mathcal{U}$. The name is given since the spectral sequence is a generalization of the Mayer-Vietoris long exact sequence for the union of two subspaces, and is thus also called the *generalized Mayer-Vietoris principle*. Note that this name is not standard. The introductory material on the construction of a spectral sequence can be found in any books on spectral sequences or homological algebra, for instance S. Mac Lane's book "*Homology*" [4], and a description of the double complex can be found for example in [2, pp. 166-168] or in [3]. For a version of the cohomology spectral sequence see [1].

The reason for writing these notes is purely expository. After searching the literature for a description of this specific spectral sequence, there was no straight forward reference with definitions and examples. I realized these notes might point the reader in the right direction if they need to use this spectral sequence. Many

---

*Date*: May 12, 2015.
2010 *Mathematics Subject Classification.* Primary 55T99.
*Key words and phrases.* Mayer-Vietoris spectral sequence, configuration space.
Partially supported by DARPA grant number N66001-11-1-4132.

details will be skipped and many proofs will be left to the reader, or a reference will be given where details or proofs can be found.

The reader is warned that these notes are far from complete, self-contained or error-free.

## 2. $FDG_{\mathbb{Z}}$-modules

Let $M$ be a differential $\mathbb{Z}$-graded module over the ring $R$ with $M = \bigoplus_{n\in\mathbb{Z}} M_n$ and a differential $d : M \to M$ of degree -1, i.e. $d(M_n) \subset M_{n-1}$ and $d^2 = 0$. If $F$ is a filtration of $M$ with

$$\cdots \subset F_{p-1}M \subset F_pM \subset F_{p+1}M \subset \cdots \subset M$$

then there is an induced filtration on the modules $M_n$ with

$$\cdots \subset F_{p-1}M_n \subset F_pM_n \subset F_{p+1}M_n \subset \cdots \subset M_n$$

which respects the differential, where $F_pM_n = F_pM \cap M_n$. The filtration $F$ induces a filtration on the graded homology module $H(M) = \{H_n(M)\}_{n\in\mathbb{Z}}$ of $M$, where $F_pH(M)$ is the image of the homology of $F_pM$ under the map induced by the inclusion of $F_pM$ into $M$ (i.e. $F_pH_q(M)$ is the image of the $q$-th homology of $F_pM$). Therefore we obtain a family of $\mathbb{Z}$-bigraded modules $\{F_pM_{p+q}\}$ called a *filtered differential $\mathbb{Z}$-graded module*, or $FDG_{\mathbb{Z}}$-module.

The filtration $F$ of $M$ is said to be *bounded* if the induced filtration of $M_n$ is finite for all $n \in \mathbb{Z}$. A spectral sequence $(E^r_{p,q}, d^r)$ is said to converge to the graded module $H = \bigoplus_{n\in\mathbb{Z}} H_n$ if there is a filtration $F$ of $H$ such that

$$E^\infty_{p,q} \cong F_pH_{p+q}/F_{p-1}H_{p+q}.$$

**Theorem 2.1.** *A filtration $F$ of a $DG_{\mathbb{Z}}$-module $M$ determines a spectral sequence $(E^r, d^r)$ with natural isomorphisms*

$$E^1_{p,q} \cong H_{p+q}(F_pM/F_{p-1}M).$$

*Moreover, if $F$ is bounded then the spectral sequence converges to $H(M)$, that is there are isomorphisms*

$$E^\infty_{p,q} \cong F_p(H_{p+q}A)/F_{p-1}(H_{p+q}A).$$

*Proof.* See [4, Chapter 16, Theorem 3.1] □

## 3. Double complexes

A *double complex* (or *bicomplex*) $N$ is a $\mathbb{Z}$-bigraded module $\{N_{p,q}\}$ with two differentials $\partial', \partial'' : N \to N$ with the properties that

$$\partial' : N_{p,q} \to N_{p-1,q}, \qquad \partial'' : N_{p,q} \to N_{p,q-1},$$

and relations

$$(\partial')^2 = 0, \qquad (\partial'')^2 = 0, \qquad \partial'\partial'' = 0.$$

The *second homology* $H''$ of $N$ is defined in the usual way by

$$H''_{p,q}(N) = ker(\partial'' : N_{p,q} \to N_{p,q-1})/im(\partial'' : N_{p,q+1} \to N_{p,q}).$$

Then there is an induced differential $\partial'$ on the bigraded second homology $H''$ and we define the homology groups $H'_pH''_q$ as follows

$$H'_pH''_q(N) = ker(\partial'' : H'_{p,q} \to H''_{p-1,q})/im(\partial' : H''_{p+1,q} \to H''_{p,q})$$

to obtain a bigraded module. Similarly, one can with defining the *first homology* $H'$ of $N$ and use the induced differential $\partial''$ to define the homology groups $H''H'(N)$.

A double complex $N$ determines a *total complex* $Tot(N)$ with

$$Tot(N)_k = \bigoplus_{p+q=k} N_{p,q}$$

with differential $\partial = \partial' + \partial''$, which makes $Tot(N)$ a differential graded module. Define the first filtration $F$ of $Tot(N)$ by

$$F_p Tot(N)_k = \bigoplus_{h \le p} N_{h,k-h}.$$

This gives the so called *first spectral sequence*.

**Theorem 3.1.** *The first spectral sequence of a double complex $N$ with associated total complex $Tot(N)$ is given by*

$$E^2_{p,q} = H'_p H''_q(N).$$

*If $N_{p,q} = 0$ for $p < 0$, then $E^2$ converges to the homology of the total complex $Tot(N)$.*

*Proof.* See [4, Chapter 16, Theorem 6.1] □

## 4. The exact couple of a $FDG_{\mathbb{Z}}$-module

An alternative way to describe a spectral sequence is via *exact couples*. Let $D$ and $E$ be $FDG_{\mathbb{Z}}$-modules. Than an *exact couple* is a pair of modules $D, E$ and three homomorphisms $i, j, k$ forming an exact triangle

$$\begin{array}{ccc} D & \xrightarrow{\;i\;} & D \\ & {}_{k}\nwarrow \quad \swarrow_{j} & \\ & E & \end{array}$$

meaning at each vertex the kernel equals the image. Set $d = jk : E \to E$ and

$$E' = ker(d)/im(d),\ D' = i(D),\ i'(a) = i(a),\ j'(ia) = [ja], k[e] = ke,$$

then it can be shown that the new derived triangle

$$\begin{array}{ccc} D' & \xrightarrow{\;i'\;} & D' \\ & {}_{k'}\nwarrow \quad \swarrow_{j'} & \\ & E' & \end{array}$$

is also exact. This is called the *derived couple* of the couple $D, E$ with the maps. Iterating this construction we get

$$\begin{array}{ccc} D^r & \xrightarrow{\;i_r\;} & D^r \\ & {}_{k_r}\nwarrow \quad \swarrow_{j_r} & \\ & E^r & \end{array}$$

a sequence of derived couples.

**Lemma 4.1.** *An exact couple of $\mathbb{Z}$-bigraded modules $D, E$ with maps of degrees*

$$deg(i) = (1, -1),\ \ deg(j) = (0, 0),\ \ deg(k) = (-1, 0),$$

*determines a spectral sequence $(E^r, d^r)$ with $d^r = j_r k_r$, for $r = 1, 2, 3 \ldots$.*

*Proof.* Note that the exact couple after $r$ iterations has

$$deg(i_r) = (1, -1), \;\; deg(j_r) = (-r+1, r-1), \;\; deg(k_r) = (-1, 0).$$

Thus $deg(d^r) = (-r, r-1)$, so each $E^{r+1}$ is the homology of $E^r$ with respect to a differential $d^r$ of the bidegree appropriate to a spectral sequence. For further details see [4, Corollary 5.3]. □

Each filtration $F$ of a $\mathbb{Z}$-graded differential module $A$ determines an exact couple as follows. The short exact sequence of complexes

$$F_{p-1}A \hookrightarrow F_pA \twoheadrightarrow F_pA/F_{p-1}A$$

gives the usual long exact sequence in homology

$$\cdots \to H_n(F_{p-1}A) \xrightarrow{i} H_n(F_pA) \xrightarrow{j} H_n(F_pA/F_{p-1}A) \xrightarrow{k} H_{n-1}(F_{p-1}A) \to \cdots$$

where $i$ is induced by the inclusion, $j$ by the projection, and $k$ is the homology connecting homomorphism. These sequences then give an exact couple with bigraded $D, E$ defined by

$$D_{p,q} = H_{p+q}(F_pA), \;\; E_{p,q} = H_{p+q}(F_pA/F_{p-1}A),$$

and the degrees of $i, j, k$ are given as above. Call this the *exact couple of the filtration $F$*.

**Theorem 4.2.** *The spectral sequence of the filtration $F$ is isomorphic to the spectral sequence of the exact couple determined by $F$.*

*Proof.* See [4, Chapter 16, Theorem 5.4] □

## 5. The double complex of a cover

Let $X$ be a simplicial complex and $\mathcal{U} = \{U_i\}_{i \in I}$ be a cover of $X$. Define a double complex $E^0 = \{E^0_{p,q}\}_{p,q \in \mathbb{Z}}$, where $E^0_{p,q}$ is the $q$-chains in a $p$-fold intersection of elements in the cover $\mathcal{U}$. That means

$$E^0_{p,q} = C_q\Big[\bigcup_{\substack{J \subseteq I \\ |J|=p}} \Big(\bigcap_{j \in J} U_j\Big)\Big],$$

together with two differential maps $d^0$ and $d^1$, where $d^0 : E^0_{p,q} \to E^0_{p,q-1}$ is the boundary map on the chains and $d^1 : E^0_{p,q} \to E^0_{p-1,q}$ explained as follows in [3]: " *For any particular $J \subseteq I$, and any particular $j' \in J$, there is an inclusion map $\cap_{j \in J} U_j \to \cap_{j \in J - \{j'\}} U_j$. If we multiply each such inclusion map with the sign of the corresponding term $J \to J - \{j'\}$ of the nerve complex boundary map and sum the maps up for all $j'$ , we get the map $d^1$.*"

Note that $(d^0)^2 = 0$, $(d^1)^2 = 0$, and $d^0 d^1 = 0$ which makes $E^0$ a double complex. Let $T_{\mathcal{U}}$ denote the total complex of $E^0$ with

$$(T_{\mathcal{U}})_n = \bigoplus_{p+q=n} E^0_{p,q}.$$

**Theorem 5.1.** *There are isomorphisms of homology groups $H_*(T_{\mathcal{U}}) \cong H_*(X)$.*

*Proof.* See [3, Theorem 1]. □

Define a filtration $F$ of $T_{\mathcal{U}}$ by setting

$$F_tT_{\mathcal{U}} = \bigoplus_{h\le t} E^0_{h,n-h}; \quad F_t(T_{\mathcal{U}})_k = \bigoplus_{h\le t} E^0_{h,k-h}.$$

Then with the filtration $F$ it follows from Theorem 3.1 that there is a spectral sequence converging to the homology of $X$.

**Theorem 5.2** (Mayer-Vietoris spectral sequence)**.** *The spectral sequence of the filtration of $T_{\mathcal{U}}$ converges to the homology of $X$.*

*Proof.* Follows directly from Theorem 5.1 and 3.1. □

## 6. An equivalent description of the spectral sequence

This description was adapted from K. Brown [2, pp. 166-168].

6.1. **Nerve of a cover.** Let $X$ be a simplicial complex and $\mathcal{U} = \{U_i\}_{i\in I}$ be a cover of $X$. The *nerve* of the cover $\mathcal{U}$, denoted $N(\mathcal{U})$ is an abstract simplicial complex defined as the collection of subsequences $\sigma \subseteq I$ with the property that $\sigma \in N(\mathcal{U})$ if and only if $X_\sigma := \cap_{i\in\sigma} U_i \neq \phi$. Such a collection clearly is an abstract simplicial complex.

6.2. **A double complex.** Let $N_p(\mathcal{U})$ be the set of $p$-simplices in $N(\mathcal{U})$. We define a chain complex $C$ with

$$C_k = \bigoplus_{\sigma\in N_k(\mathcal{U})} C(X_\sigma) = \bigoplus_{\sigma\in N_k(\mathcal{U})} C(\cap_{i\in\sigma} U_i)$$

and with boundary map $\partial : C_k \to C_{k-1}$ given by

$$\partial\sigma = \partial\{j_0 < \cdots < j_k\} = \sum_{i=0}^{k} (-1)^i \{j_0 < \cdots < \widehat{j_i} < \cdots < j_k\}$$

extending linearly over the direct sum. Now define a double complex $C$ with

$$C_{pq} = \bigoplus_{\sigma\in N_p(\mathcal{U})} C_q(X_\sigma) = \bigoplus_{\sigma\in N_p(\mathcal{U})} C_q(\cap_{i\in\sigma} U_i)$$

and the boundary map $\partial' : C_{p,q} \to C_{p,q-1}$ the standard boundary map in simplicial homology.

Let $E^1_{pq} = H_q(C_p) = \bigoplus_{\sigma\in N_p(\mathcal{U})} H_q(X_\sigma)$ be homology of the double complex $C$ with respect to the boundary $\partial$. There is an induced boundary map $\partial'$ on $E^1$ with homology the spectral sequence

$$E^2_{pq} = H_p(E^1_{pq}) = H_p\Big(\bigoplus_{\sigma\in N_p(\mathcal{U})} H_q(X_\sigma)\Big).$$

It can be shown that $E^2_{pq} \Rightarrow H_{p+q}(X)$. This is called the *Mayer-Vietoris spectral sequence*, a name which is not entirely standard.

## 7. An example with configuration spaces

The example that follows is most probably not the most enlightening example, but gives a nice application of this spectral sequence. For this example the reader can also look at [5, Section 3.1].

Let $R_n$ be the connected tree with $n$ edges and with exactly one vertex with valence $n$ and the other vertices of valence 1, see Figure 1.

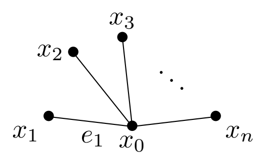


Figure 1. $R_4$, $\deg(x_0) = n$

Note that for $n \geq 3$ we can write $R_n = R_{n-1} \vee_{x_0} e_1$, where $e_1$ is the first edge in the figure and $R_{n-1}$ is the union of the other edges. We assume that $n \geq 3$ since for $n = 2$ we get the unit interval, which is well understood in the following construction.

Recall that the space of ordered $k$-configurations of a topological space $X$ is defined by

$$Conf(X, n) = \{(t_1, \dots, t_k) \in X^k : x_i \neq x_j \text{ if } i \neq j\}.$$

There is a cover of the space of ordered 2-configurations $Conf(R_n, 2)$, given by $U = \{U_{11}, U_{12}, U_{21}, U_{22}\}$, where

$$U_{11} = Conf(e_1, 2), \qquad U_{12} = e_1 \times R_{n-1} - \{(x_0, x_0)\},$$
$$U_{22} = Conf(R_{n-2}, 2), \qquad U_{21} = R_{n-1} \times e_1 - \{(x_0, x_0)\}.$$

Consider the intersection poset $P_U$ of the cover, that is the poset consisting of all the elements of $U$ and their inclusions partially ordered by inclusion. One can check that all the inclusions are cofibrations. Moreover, by inspection we get the following lemma.

**Lemma 7.1.** *The elements in the cover $U$ satisfy the following:*

(1) $e_1 - \{x_0\} \simeq *$,
(2) $R_{n-1} - \{x_0\} \simeq \{*_1, \dots, *_n\}$,
(3) $U_{11} \simeq \{*_1, *_2\}$,
(4) $U_{12} \simeq U_{21} \simeq *$,
(5) $U_{11} \cap U_{12} \simeq U_{11} \cap U_{21} \simeq *$,
(6) $U_{12} \cap U_{22} \simeq U_{21} \cap U_{22} \simeq \{*_1, \dots, *_{n-1}\}$.

*Proof.* Proof is left an exercise, or see [5, Section 3.1]. □

Hence, the Mayer-Vietoris spectral sequence for $Conf(R_n, 2)$ and $P_U$ has the following properties. Recall the definition of the fist page of this spectral sequence from the preceding discussion. Thus the first page $E^1$ of the homology spectral sequence is given by:

(1) $E^1_{0,0} = H_0(U_{11}) \oplus H_0(U_{12}) \oplus H_0(U_{21}) \oplus H_0(U_{22}) \cong (\oplus_4 \mathbb{Z}) \oplus H_0(Conf(R_{n-1}, 2)$,

$$
\begin{array}{c|ccccc}
\vdots & \vdots & & 0 & 0 \\
1 & H_1(Conf(R_{n-1}),2) & & 0 & 0 \\
0 & \bigoplus_4 \mathbb{Z} \oplus H_0 & \longleftarrow & \bigoplus_{2n} \mathbb{Z} & 0 \\
\hline
 & 0 & & 1 & 2
\end{array}
$$

FIGURE 2. $E^1$ page of the Mayer-Vietoris spectral sequence

(2) $E^1_{1,0} = H_0(U_{11} \cap U_{12}) \oplus H_0(U_{11} \cap U_{21}) \oplus H_0(U_{12} \cap U_{22}) \oplus H_0(U_{21} \cap U_{22}) \cong \oplus_{2n}\mathbb{Z}$,
(3) $E^1_{0,q} = H_q(Conf(R_{n-1}, 2)$ for $q \geq 1$,
(4) $E^1_{p,q} = 0$ otherwise.

There will be a difference in the treatment of this spectral sequence depending on the value of the number of edges $n$. Recall that the differential $d^r_{p,q}$ is a map

$$d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}.$$

When $n = 3$ the only possibly nonzero differential has the following image and kernel.

**Lemma 7.2.** *If $n = 3$ then $Im(d^1_{1,0}) \cong \bigoplus_5 \mathbb{Z}$ and $Ker(d^1_{1,0}) \cong \mathbb{Z}$.*

*Proof.* Note that the configuration space $Conf(R_2, 2) \simeq \{*_1, *_2\}$. Assume that the homology groups, which are free abelian groups, have the following generators:

$$
\begin{aligned}
H_0(U_{11}) &= \langle f_1, f_2 \rangle, & H_0(U_{12}) &= \langle h_1 \rangle, \\
H_0(U_{22}) &= \langle g_1, g_2 \rangle, & H_0(U_{21}) &= \langle h_2 \rangle, \\
H_0(U_{11} \cap U_{12}) &= \langle a_1 \rangle, & H_0(U_{12} \cap U_{22}) &= \langle b_1, b_2 \rangle, \\
H_0(U_{11} \cap U_{21}) &= \langle a_2 \rangle, & H_0(U_{21} \cap U_{22}) &= \langle c_1, c_2 \rangle.
\end{aligned}
$$

Notice that the differential is induced by the inclusion maps of the intersection in the poset $P_U$. One can check the following:

$$
\begin{aligned}
d^1_{1,0}(a_1) &= f_1 - h_1, & d^1_{1,0}(a_2) &= f_2 - h_2, \\
d^1_{1,0}(b_1) &= h_1 - g_1, & d^1_{1,0}(b_2) &= h_2 - g_2, \\
d^1_{1,0}(c_1) &= h_2 - g_1, & d^1_{1,0}(c_2) &= h_2 - g_2.
\end{aligned}
$$

Therefore, the image of $d^1_{1,0}$ is generated by

$$\{f_1 - h_1, f_2 - h_2, h_1 - g_1, h_2 - g_2, h_2 - g_1, h_2 - g_2\},$$

which has dimension 5. Finally the kernel has dimension 1. □

Hence, we have that $E^2_{0,0} \cong Ker(d^1_{1,0})/Im(d^1_{1,0}) \cong \mathbb{Z}$, and $Conf(R_3, 2)$ is path connected. By an induction hypothesis it follows that $Conf(R_n, 2)$ is path connected for all $n \geq 3$.

**Lemma 7.3.** *If $n \geq 4$ then $Im(d^1_{1,0}) \cong \bigoplus_4 \mathbb{Z}$ and $Ker(d^1_{1,0}) \cong \bigoplus_{2(n-2)} \mathbb{Z}$.*

*Proof.* This is almost the same as the previous proof. Assume the configuration space $Conf(R_n, 2)$ is connected for $n \geq 3$ Assume that the homology groups of the poset, which are free abelian groups, have the following generators:

$$\begin{array}{ll} H_0(U_{11}) = \langle f_1, f_2\rangle, & H_0(U_{12}) = \langle h_1\rangle, \\ H_0(U_{22}) = \langle h_3\rangle, & H_0(U_{21}) = \langle h_2\rangle, \\ H_0(U_{11} \cap U12) = \langle a_1\rangle, & H_0(U_{12} \cap U_{22}) = \langle b_1, \dots, b_{n-1}\rangle, \\ H_0(U_{11} \cap U21) = \langle a_2\rangle, & H_0(U_{21} \cap U_{22}) = \langle c_1, \dots, c_{n-1}\rangle. \end{array}$$

Notice that the differential is induced by the inclusion maps of the intersection in the poset $P_U$. One can check the following:

$$\begin{array}{ll} d^1_{1,0}(a_1) = f_1 - h_1, & d^1_{1,0}(a_2) = f_2 - h_2, \\ d^1_{1,0}(b_i) = h_1 - h_3, \text{ for all } i, & d^1_{1,0}(c_i) = h_2 - h_3, \text{ for all } i. \end{array}$$

Therefore, the image of $d^1_{1,0}$ is generated by

$$\{f_1 - h_1, f_2 - h_2, h_1 - h_3, h_2 - h_3\},$$

which has dimension 4. Finally the kernel has dimension $2n - 4 = 2(n - 2)$. □

It follows from the spectral sequence that if $n \geq 3$, then

(1) $H_0(Conf(R_n, 2); \mathbb{Z}) = \mathbb{Z}$,
(2) $H_1(Conf(R_n, 2); \mathbb{Z}) = \bigoplus_{2(n-2)} \mathbb{Z} \oplus H_1(Conf(R_{n-1}, 2))$ and
(3) $H_k(Conf(R_n, 2) = H_1(Conf(R_{n-1}, 2)$.

**Theorem 7.4.** *If $n \geq 3$ The homology of $Conf(R_n, 2)$ is given by*

$$H_k(Conf(R_n, 2); \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \bigoplus_{(n-1)(n-2)-1} \mathbb{Z} & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases} \tag{7.1}$$

*Proof.* Follows from Lemmas 7.2 and 7.3, and the iterations:

$$\begin{aligned} H_1(Conf(R_n, 2); \mathbb{Z}) &= \bigoplus_{2(n-2)} \mathbb{Z} \oplus H_1(Conf(R_{n-1}, 2)) \\ &= \bigoplus_{2(n-2)+2(n-3)} \mathbb{Z} \oplus H_1(Conf(R_{n-2}, 2)) \\ &= \cdots = \bigoplus_{(n-1)(n-2)-1} \mathbb{Z}, \end{aligned}$$

and for $k \geq 2$

$$\begin{aligned} H_k(Conf(R_n, 2) &= H_k(Conf(R_{n-1}, 2) = H_k(Conf(R_{n-2}, 2) \\ &= \cdots = H_k(Conf(R_2, 2) = 0. \end{aligned}$$

□

## References

[1] Bott, R. and Tu, L. W. Differential forms in algebraic topology. *Graduate texts in Mathematics 82* (1982).
[2] Brown, K. S. *Cohomology of groups.* GTM 87. Springer-Verlag, New York, 1982.
[3] Lipsky, D., Skraba, P., and Vejdemo-Johansson, M. A spectral sequence for parallelized persistence. *arXiv:1112.1245* (2011).
[4] Mac Lane, S. *Homology.* Classics in Mathematics. Springer, 1995.
[5] Sun, Q. *Configuration Spaces of Singular Spaces.* PhD thesis, University of Rochester, 2011.

Department of Mathematics, Tulane University, New Orleans, LA 70118
*E-mail address*: mstafa@tulane.edu