Applicable Analysis
Vol. 90, No. 1, January 2011, 85–102

Cauchy convergence schemes for some nonlinear partial differential equations†

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Communicated by Y. Xu

(*Received 4 February 2010; final version received 21 February 2010*)

Motivated by ongoing work in the theory of stochastic partial differential equations we develop direct methods to infer that the Galerkin approximations of certain nonlinear partial differential equations are Cauchy (and therefore convergent). We develop such a result for the Navier–Stokes equations in space dimensions two and three, and for the primitive equations in space dimension two. The analysis requires novel estimates for the nonlinear portion of these equations and delicate interpolation results concerning subspaces.

Keywords: nonlinear PDE; primitive equations; Navier–Stokes; Galerkin approximations; compactness methods; fractional spaces; interpolation; comparison estimates; stochastic PDE

AMS Subject Classifications: 35Q86; 35Q30; 46B70

1. Introduction

Galerkin approximations are commonly used to prove the existence of solutions of (linear and) nonlinear evolution equations. After constructing such approximations, one derives *a priori* estimates for these solutions. These *a priori* estimates produce weak convergence results which are sufficient to pass to the limit in the linear case. In the nonlinear case, one may pass to the limit by applying a classical compactness criterion [1,2]. In any case by proving that, in certain spaces, the Galerkin approximations are (sub-sequentially) convergent we implicitly deduce that they are Cauchy in the corresponding spaces.

In this article, we tackle the following unusual question, namely to prove *directly* that the Galerkin solutions of certain nonlinear equations are Cauchy. Of course we do not avoid the (classical) part of convergence concerning the *a priori* estimates. Besides its intrinsic interest, the question that we address is motivated by ongoing work, [3,4], concerning nonlinear stochastic partial differential equations (SPDEs).

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†Dedicated to Professor Vsevolod Alekseevich Solonnikov on the occasion of his 75th birthday.

ISSN 0003–6811 print/ISSN 1563–504X online

DOI: 10.1080/00036811003735956
http://www.informaworld.com

For such systems, particularly those that are forced by nonlinear white noise driven terms, the *a priori* estimates may not be carried out 'pathwise', that is ω by ω, in the underlying probability space. The classical compactness methods are therefore complicated by the addition of stochastic terms and novel approaches are required in order to pass to the limit. In the aforementioned articles [3,4], Cauchy convergence is established to overcome this difficulty. On this point the approach is very similar to the deterministic case. As such we thought it was worthwhile to separate this analysis which has some intricate elements.

In what follows, and in parallel to the aforementioned articles [3,4], we prove the Cauchy convergence of the Galerkin approximations for the 2D primitive equations (PEs) and for the Navier–Stokes equations in space dimensions 2 and 3. The result that we obtain is local in time for the PEs and 3D Navier–Stokes equations and global in time for the 2D Navier–Stokes Equations.

The analysis below is organized as follows: in Section 2, we recall the 2D PEs and their mathematical setting. In Section 3, we establish the Cauchy convergence. Due to the discretization of the basic equations, stray elements depending on only one of the orders appear for these estimates. We address the magnitude of these terms by making use of the generalized Poincaré inequality. The analysis for this point requires both novel estimates on the nonlinear portion of the PEs H^1 (presented in Lemma 2.1), and the usage of fractional order spaces to avoid difficulties with the boundary. Section 4 addresses this point and makes extensive application of delicate interpolation results from [5]. Section 5 establishes similar results for the Navier–Stokes equations.

2. Review of the 2D PEs and their mathematical setting

The PEs are widely regarded as a fundamental description of geophysical scale fluid flow. They provide the analytical core of large general circulation models (GCMs) that are at the forefront of numerical simulations of the earth's oceans and atmosphere (see e.g. [6]). Beyond their significance in geophysics, these systems have an intricate analytical structure that has led to a large body of work in the mathematics community. See [7–15] and the recent surveys [16,17].

The two-dimensional version of these equations takes the form:

$$\partial_t u + u\partial_x u + w(u)\partial_z u - \nu\Delta u - fv + \partial_x p_s - \beta_T g \rho_0 \int_z^0 \partial_x T \, d\bar{z} = F_u + \sigma_u(\mathbf{v}, T)\dot{W}_1, \tag{2.1a}$$

$$\partial_t v + u\partial_x v + w(u)\partial_z v - \nu\Delta v + fu = F_v + \sigma_v(\mathbf{v}, T)\dot{W}_2, \tag{2.1b}$$

$$w(u) = \int_z^0 \partial_x u \, d\bar{z}, \qquad \int_{-h}^0 u \, dz = 0, \tag{2.1c}$$

$$\partial_t T + u\partial_x T + w(u)\partial_z T - \mu\Delta T = F_T + \sigma_T(\mathbf{v}, T)\dot{W}_3. \tag{2.1d}$$

Here $(\mathbf{v}, w) = (u, v, w)$, T denote, respectively, the flow field and the temperature of the fluid being modelled, $\mathbf{v}$ is the horizontal velocity. The coefficients ν, μ account for the molecular viscosity and the rate of heat diffusion, g is the gravitational constant and ρ_0

represents the mean density of the fluid; f, which is a function of the earth's rotation and the local latitude is taken constant here for simplicity. The terms F_u, F_v and F_T correspond to external sources of horizontal momentum and heat. They vanish in the ocean (and F_u, F_v in the atmosphere); they are added for mathematical generality or to consider nonhomogeneous boundary conditions (which we do not do here).

We consider the evolution of (2.1) over a rectangular domain $\mathcal{M}=(0,L)\times(-h,0)$ and label the parts of the boundary $\Gamma_i=(0,L)\times\{0\}$, $\Gamma_b=(0,L)\times\{-h\}$ and $\Gamma_l=\{0,L\}\times(-h,0)$. We posit the physically realistic boundary conditions

$$\partial_z\mathbf{v}+\alpha_{\mathbf{v}}\mathbf{v}=0,\quad w=0,\quad \partial_z T+\alpha_T T=0,\quad \text{on }\Gamma_i, \tag{2.2a}$$

$$\mathbf{v}=0,\quad \partial_x T=0,\quad \text{on }\Gamma_l, \tag{2.2b}$$

$$\mathbf{v}=0,\quad w=0,\quad \partial_z T=0,\quad \text{on }\Gamma_b. \tag{2.2c}$$

The equations and boundary conditions (2.1), (2.2) are supplemented by initial conditions for u, v and T, that is

$$u=u_0,\quad v=v_0,\quad T=T_0,\quad \text{at } t=0. \tag{2.3}$$

For a detailed treatment of the physical derivation and significance of (2.1), (2.2) see [18,19].

We next review the mathematical setting for (2.1)–(2.3). For the most part our presentation and notations follow the recent survey [16], to which we refer the reader for a more detailed treatment.

The main function spaces used are defined as follows. Take:

$$H:=\left\{U=(u,v,T)\in L^2(\mathcal{M})^3:\int_{-h}^0 u\,\mathrm{d}z=0\right\}.$$

We equip H with the inner product[1]

$$(U,U^\sharp):=\int_{\mathcal{M}}\mathbf{v}\cdot\mathbf{v}^\sharp\mathrm{d}\mathcal{M}+\int_{\mathcal{M}}TT^\sharp\mathrm{d}\mathcal{M},\quad U=(\mathbf{v},T),\quad U^\sharp=(\mathbf{v}^\sharp,T^\sharp).$$

Here and below we shall make use of the vertical averaging operator $P\phi=\frac{1}{h}\int_{-h}^0\phi(\bar z)\mathrm{d}\bar z$ and its orthogonal complement $Q\phi=\phi-P\phi$. Note that the projection operator $\Pi: L^2(\mathcal{M})^3\to H$ may be explicitly written according to $U\mapsto(Qu,v,T)$. We also define

$$V:=\left\{U=(u,v,T)\in H^1(\mathcal{M})^3:\int_{-h}^0 u\,\mathrm{d}z=0,\mathbf{v}=0\text{ on }\Gamma_l\cup\Gamma_b\right\}.$$

Here we take the inner product $((\cdot,\cdot))=\nu((\cdot,\cdot))_1+\mu((\cdot,\cdot))_2$ where for given $U=(\mathbf{v},T)$, $U^\sharp=(\mathbf{v}^\sharp,T^\sharp)\in V$,

$$((U,U^\sharp))_1:=\int_{\mathcal{M}}\partial_x\mathbf{v}\cdot\partial_x\mathbf{v}^\sharp+\partial_z\mathbf{v}\cdot\partial_z\mathbf{v}^\sharp\,\mathrm{d}\mathcal{M}+\alpha_{\mathbf{v}}\int_{\Gamma_i}\mathbf{v}\cdot\mathbf{v}^\sharp\,\mathrm{d}x,$$

$$((U,U^\sharp))_2:=\int_{\mathcal{M}}\partial_x T\partial_x T^\sharp+\partial_z T\partial_z T^\sharp\,\mathrm{d}\mathcal{M}+\alpha_T\int_{\Gamma_i}TT^\sharp\,\mathrm{d}x.$$

Note that under these definitions a Poincaré type inequality $|U| \leq C\|U\|$ holds for all $U \in H^1(\mathcal{M})^3 \supset V$. Moreover the norms $\|\cdot\|_{H^1}$, $\|\cdot\|$ may be seen to be equivalent over all of $H^1(\mathcal{M})^3$.

Even if U is very regular, several terms in the abstract formulation of (2.1) do not belong to V (see (2.11), (2.14), (2.15) below). As such, we shall also make use of the additional auxiliary spaces:

$$\tilde{V} := \left\{ U = (u, v, T) \in H^1(\mathcal{M})^3 : \int_{-h}^{0} u \, \mathrm{d}z = 0, \mathbf{v} = 0 \text{ on } \Gamma_l \right\},$$
$$\mathcal{Z} := \left\{ U = (u, v, T) \in H^1(\mathcal{M})^3 : \mathbf{v} = 0 \text{ on } \Gamma_l \right\}.$$

As for V we endow both spaces with the norm $\|\cdot\|$. One may verify that Π also maps Z onto $\tilde{V}$ and is continuous on $H^1(\mathcal{M})^3$.

Some intermediate order space are employed in the analysis below. Following the notation of [5, Chapter 1, Section 2.1] we define, for $s \in [0, 1]$,

$$\tilde{V}_s = [\tilde{V}, H]_{1-s}, \quad V_s = [V, H]_{1-s}. \tag{2.4}$$

Proposition 2.1 establishes the result important for our analysis that $V_s = \tilde{V}_s$ for $s \in (0, 1/2)$.

Finally we take $V_{(2)} = H^2(\mathcal{M})^3 \cap V$ and equip this space with the classical $H^2(\mathcal{M})$ norm which we denote by $|\cdot|_{(2)}$.

The linear second-order terms in the equation are captured in the Stokes-type operator A which is understood as a bounded operator from V to V' via $\langle AU, U^\sharp \rangle = ((U, U^\sharp))$. The additional terms in the variational formulation of this portion of the equation term capture the Robin boundary condition (2.2a). They may be formally derived by multiplying $-\nu\Delta u$, $-\nu\Delta v$, $-\mu\Delta T$ in (2.1a), (2.1b), (2.1d) by test functions $u^\sharp$, $v^\sharp$, $T^\sharp$, integrating over $\mathcal{M}$ and integrating by parts. We shall make use of the subspace $D(A) \subset V_{(2)}$ given by

$$D(A) = \{U = (\mathbf{v}, T) \in V_{(2)} : \partial_z \mathbf{v} + \alpha_{\mathbf{v}} \mathbf{v} = 0, \partial_z T + \alpha_T T = 0 \text{ on } \Gamma_i,$$
$$\partial_x T = 0 \text{ on } \Gamma_l, \partial_z T = 0 \text{ on } \Gamma_b, \}.$$

On this space we may extend A to an unbounded operator by defining

$$AU = \begin{pmatrix} -\nu Q \Delta u \\ -\nu \Delta v \\ -\mu \Delta T \end{pmatrix}, \quad U \in D(A). \tag{2.5}$$

Since A is self adjoint, with a compact inverse $A^{-1} : H \to D(A)$ we may apply the standard theory of compact, symmetric operators to guarantee the existence of an orthonormal basis $\{\Phi_k\}_{k \geq 0}$ for H of eigenfunctions of A with the associated eigenvalues $\{\lambda_k\}_{k \geq 0}$ forming an unbounded, increasing sequence. Note that by the regularity results in [20] or [21] we have $\Phi_k \in D(A) \subset V_{(2)}$. Define

$$H_n = span\{\Phi_1, \ldots, \Phi_n\}.$$

Take P_n and $Q_n = I - P_n$ to be the projections from H onto H_n and its orthogonal complement, respectively. For all $m > n$ we let $P_m^n = P_m - P_n$.

We shall also make use of the fractional powers of A. Given $\alpha > 0$, take

$$D(A^\alpha) = \left\{ U \in H : \sum_k \lambda_k^{2\alpha} |U_k|^2 < \infty \right\}, \tag{2.6}$$

where $U_k = (U, \Phi_k)$. On this set we may define A^α according to

$$A^\alpha U = \sum_k \lambda_k^\alpha U_k \Phi_k, \quad \text{for } U = \sum_k U_k \Phi_k. \tag{2.7}$$

Accordingly we equip $V_\alpha = D(A^{\alpha/2})$ with the norm

$$|U|_\alpha = |A^{\alpha/2} U| = \left(\sum_k \lambda_k^\alpha |U_k|^2 \right)^{1/2}. \tag{2.8}$$

It is direct to verify that $D(A^{1/2}) = V$ and that the associated norms $\|\cdot\|$ and $|\cdot|_1$ are identical. Moreover, by exploiting the regularity theory developed in [20], we may also infer the equivalence of $|\cdot|_{(2)}$ and $|\cdot|_2$ on $D(A)$. Classically, we have the generalized (and reverse) Poincaré estimates

$$|P_n U|_{\alpha_2}^2 \le \lambda_n^{\alpha_2 - \alpha_1} |P_n U|_{\alpha_1}^2, \quad |Q_n U|_{\alpha_1}^2 \le \frac{1}{\lambda_n^{\alpha_2 - \alpha_1}} |Q_n U|_{\alpha_2}^2, \tag{2.9}$$

for any $\alpha_1 < \alpha_2$.

The following result, whose proof we postpone to Section 4, is used in Theorem 3.1.

Proposition 2.1 *Let $\alpha \in (0, 1/2)$. Then $V_\alpha = \tilde{V}_\alpha = D(A^{\alpha/2})$ and therefore the estimate*

$$|U|_\alpha \le |U|^{1-\alpha} \|U\|^\alpha \tag{2.10}$$

holds for all $U \in \tilde{V}$

Note that in some previous works, the second component of the pressure (cf [16, Section 2]) is included in the definition of the principal linear operator A. Since this breaks the symmetry of A we relegate such terms to a separate, lower order operator $A_p : V \to H$ defined by

$$A_p U = \begin{pmatrix} -Q \int_z^0 \partial_x T \, d\bar{z} \\ 0 \\ 0 \end{pmatrix}. \tag{2.11}$$

If $U \in D(A)$, $A_p U \in \tilde{V}$ and we have the estimates

$$|A_p U| \le c \|U\|, \quad \|A_p U\| \le c |U|_{(2)}. \tag{2.12}$$

We next capture the nonlinear portion of (2.1). In accordance with (2.1c) and (2.2) ($w = 0$ on Γ_i) we *define* the diagnostic function:

$$w(U) = w(u) = \int_z^0 \partial_x u \, d\bar{z}, \quad U = (u, v, T) \in V. \tag{2.13}$$

For $U = (\mathbf{v}, T) \in V$, $U^\sharp = (\mathbf{v}^\sharp, T^\sharp) \in V_{(2)}$ we take $B(U, U^\sharp) = B_1(U, U^\sharp) + B_2(U, U^\sharp)$ where

$$B_1(U, U^\sharp) := \begin{pmatrix} Q(u \partial_x u^\sharp) \\ u \partial_x v^\sharp \\ u \partial_x T^\sharp \end{pmatrix} = \begin{pmatrix} B_1^1(u, u^\sharp) \\ B_1^2(u, v^\sharp) \\ B_1^3(u, T^\sharp) \end{pmatrix} \tag{2.14}$$

and

$$B_2(U, U^\sharp) := \begin{pmatrix} Q(w(u)\partial_z u^\sharp) \\ w(u)\partial_z v^\sharp \\ w(u)\partial_z T^\sharp \end{pmatrix} = \begin{pmatrix} B_2^1(u, u^\sharp) \\ B_2^2(u, v^\sharp) \\ B_2^3(u, T^\sharp) \end{pmatrix}. \tag{2.15}$$

The following lemma summarizes some properties of B needed in the sequel.[2]

LEMMA 2.1 *B is well-defined as a bilinear continuous map from $V \times V_{(2)}$ into H and:*

(i) *for any $U \in V$, $U^\sharp \in V_{(2)}$ and $U^\flat \in H$*

$$|\langle B(U, U^\sharp), U^\flat \rangle| \le c\|U\| \|U^\sharp\|^{1/2} |U^\sharp|_{(2)}^{1/2} |U^\flat|. \tag{2.16}$$

In particular, for $U \in V_{(2)}$,

$$|B(U)|^2 \le \|U\|^3 |U|_{(2)}; \tag{2.17}$$

(ii) *for $U \in V_{(2)}$, $B(U) \in \tilde{V} \subset \tilde{V}_{1/4} = V_{1/4} = D(A^{1/8})$ and satisfies the estimate*

$$\|B(U)\|^2 \le c\|U\| |U|_{(2)}^3. \tag{2.18}$$

Proof The estimate (2.16) may be established with the now standard anisotropic type techniques. See [21] or [14]. To the best of our knowledge, the H^1 estimate for B given in (ii) is new. To prove it, we observe that, for $U = (u, v, T) \in V_{(2)}$, $B(U) = \Pi\mathcal{B}(U)$ where

$$\mathcal{B}(U) = \begin{pmatrix} u\partial_x u + w\partial_z u \\ u\partial_x v + w\partial_z v \\ u\partial_x T + w\partial_z T \end{pmatrix}, \tag{2.19}$$

and Π is the orthogonal projector from $L^2(\mathcal{M})^3$ onto H. Note that Π is also continuous from $H^1((\mathcal{M}))^3$ into itself (see e.g. [22] for the similar but more involved result for the Navier-Stokes equations). Hence it suffices, to prove (2.17), to show that $\|\mathcal{B}(U)\|$ is bounded by the right-hand side of (2.17) for a suitable constant c.

To estimate the L^2-norm of the gradient of $\mathcal{B}(U)$ we ought to estimate the L^2-norm of terms like

$$u\partial^2 u,\ (\partial u)^2,\ (\partial w)(\partial u),\ w\partial^2 u, \tag{2.20}$$

where $\partial = \partial_x$ or ∂_z (i.e. $\partial = \partial/\partial x$ or $\partial/\partial z$). The most delicate terms are terms like $(\partial w)(\partial u)$ and $w\,\partial^2 u$ that we estimate using the anisotropic inequalities as in [16,21]. Hence for e.g. $(\partial w)(\partial u)$:

$$\begin{aligned} |\partial w \partial u|_{L^2(\mathcal{M})} &= \int_{\mathcal{M}} (\partial w)^2 (\partial u)^2 \,\mathrm{d}\mathcal{M} \\ &\le \int_0^L |\partial w|_{L_z^\infty}^2 |\partial u|_{L_z^2}^2 \mathrm{d}x \\ &\le \left(\int_0^L |\partial w|_{L_z^\infty}^2 \mathrm{d}x \right) |\partial u|_{L_z^2 L_x^\infty}^2. \end{aligned}$$

Using the expression (2.13) of $w = w(u)$, for respectively $\partial = \partial_z$, ∂_x, we bound the first term above by $|\partial_x u|_{L^2(\mathcal{M})}|\partial_{xz}u|_{L^2(\mathcal{M})}$ or by $h|\partial_{xx}u|^2_{L^2(\mathcal{M})}$, which in both cases are dominated by $|U|^2_{(2)}$.

The term $|\partial u|^2_{L^2_z L^\infty_2}$ is estimated as follows [16,21]:

$$|\partial u|_{L^2_z L^\infty_x} = |\theta|_{L^\infty_x} \leq c|\theta|^{1/2}_{L^2_x}\ |\partial_x \theta|^{1/2}_{L^2_x}, \qquad \text{where } \theta = \left(\int_{-h}^0 |\partial u|^2 \, \mathrm{d}\bar{z}\right)^{1/2}.$$

Hence, pointwise,

$$\partial_x \theta = \frac{1}{\theta}\int_{-h}^0 \partial u \partial_x \partial u \, \mathrm{d}\bar{z} \Rightarrow |\partial_x \theta| \leq \left(\int_h^0 |\partial_x \partial u|^2 \, \mathrm{d}\bar{z}\right)^{1/2},$$

and

$$|\partial_x \theta|_{L^2_x} \leq |\partial^2 u|_{L^2(\mathcal{M})}.$$

In the end

$$|\partial u|_{L^2_z L^\infty_\infty} \leq c|\partial u|^{1/2}_{L^2(\mathcal{M})}|\partial^2 u|^{1/2}_{L^2(\mathcal{M})} \leq c\|U\|^{1/2}|U|^{1/2}_{(2)},$$

and $|\partial w \partial u|_{L^2(\mathcal{M})}$ is bounded by a term like the right-hand side of (2.17).

The proof is similar for a term like $w\partial^2 u$, and, as we said, the other terms are easier to handle. ■

We capture the Coriolis forcing with the bounded operator $E : H \to H$ given by

$$EU := \begin{pmatrix} -Qfv \\ fu \\ 0 \end{pmatrix}. \tag{2.21}$$

We observe that E is also continuous from V to $\tilde{V}$ and that

$$|EU| \leq f|U|, \qquad \|EU\| \leq f\|U\|. \tag{2.22}$$

For the external forcing terms F_u, F_v, F_T we let:

$$F = \begin{pmatrix} QF_u \\ F_v \\ F_T \end{pmatrix}.$$

For Theorem 3.1 below we assume $F \in L^2_{loc}([0,\infty), H)$ and that $U_0 = (u_0, v_0, T) \in V$. With the definitions (2.5), (2.11), (2.14), (2.15) and (2.21) we may reformulate (2.1) as an abstract equation:

$$\frac{\mathrm{d}U}{\mathrm{d}t} + AU + A_p U + B(U) + EU = F, \qquad U(0) = U_0. \tag{2.23}$$

We finally recall the definition of the Galerkin approximations associated with (2.23). We say that, $U^{(n)} \in C([0,\infty); H_n)$ is a solution of the Galerkin system of order n if:

$$\begin{aligned} &\frac{\mathrm{d}U^{(n)}}{\mathrm{d}t} + AU^{(n)} + P_n(A_p U^{(n)} + B(U^{(n)}) + EU^{(n)}) = P_n F \\ &U^{(n)}(0) = P_n U_0. \end{aligned} \tag{2.24}$$

■

3. Comparison estimates for the Galerkin system

In this section, we establish that the solutions of (2.24) are 'locally Cauchy' in strong type spaces. The result (and proof) may be seen as an alternative to the classical Aubin compactness theorem [1] although this is not our primary motivation. Note in this connection that we infer the convergences of the Galerkin approximations without any estimates on $\mathrm{d}U^{(n)}/\mathrm{d}t$.

Theorem 3.1 *Suppose that* $F \in L^2_{loc}([0,\infty), H)$, $U_0 \in V$. *Then there exists* $t'_* > 0$ *independent of n and depending only on the data, such that,* $\{U^{(n)}\}_{n\geq 1}$, *the solutions of* (2.24), *are Cauchy* (*and hence convergent*) *in* $C([0,t'_*]; V) \cap L^2([0,t'_*]; D(A))$

Proof By taking the inner product of (2.24) with $AU^{(n)}$ and applying (2.12), (2.16), (2.22) we classically infer that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|U^{(n)}\|^2 + |AU^{(n)}|^2 \leq c_1(1 + |P_nF|^2 + \|U^{(n)}\|^6). \tag{3.1}$$

With $\|U^{(n)}(0)\| = \|P_nU_0\| \leq \|U_0\|$ and $|P_nF| \leq |F|$, we infer from (3.1) that $1 + \|U^{(n)}(t)\|^2 \leq 2(1 + \|U_0\|^2)$ for an initial period during which

$$\frac{\mathrm{d}}{\mathrm{d}t}\|U^{(n)}\|^2 \leq c_1(|F|^2 + (1 + \|U^{(n)}\|^2)^3),$$

so that

$$1 + \|U^{(n)}(t)\|^2 \leq 1 + \|U_0\|^2 + c_1\int_0^t |F(s)|^2\,\mathrm{d}s + 2^3 c_1 t(1 + \|U_0\|^2)^3,$$

and thus $1 + \|U^{(n)}(t)\|^2 \leq 2(1 + \|U_0\|^2)$ holds until the time t'_* where

$$c_1\int_0^{t'_*} |F(s)|^2\,\mathrm{d}s + 2^3 c_1 t'_*(1 + \|U_0\|^2)^3 = 1 + \|U_0\|^2. \tag{3.2}$$

Returning to (3.1) we obtain the classical *a priori* estimates which we write in the form:

$$\sup_n\left(\sup_{t\in[0,t'_*]}\|U^{(n)}\|^2 + \int_0^{t'_*}|AU^{(n)}|^2\,\mathrm{d}t\right) < \infty. \tag{3.3}$$

Fix $m > n$ and denote $R^{(m,n)} = U^{(m)} - U^{(n)}$. Subtracting the equations for $U^{(m)}$ and $U^{(n)}$ and taking an inner product with $AR^{(m,n)}$ gives

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}&\|R^{(m,n)}\|^2 + |AR^{(m,n)}|^2\\ &= -\langle P_mB(U^{(m)}) - P_nB(U^{(n)}), AR^{(m,n)}\rangle\\ &\quad - \langle P_mA_pU^{(m)} - P_nA_pU^{(n)}, AR^{(m,n)}\rangle\\ &\quad - \langle P_mEU^{(m)} - P_nEU^{(n)}, AR^{(m,n)}\rangle\\ &\quad + \langle P^n_mF, AR^{(m,n)}\rangle.\end{aligned} \tag{3.4}$$

We consider first the nonlinear term:

$$\begin{aligned}&\langle P_mB(U^{(m)}) - P_nB(U^{(n)}), AR^{(m,n)}\rangle\\ &= \langle B(R^{(m,n)}, U^{(m)}) + B(U^{(n)}, R^{(m,n)}) + P^n_mB(U^{(n)}), AR^{(m,n)}\rangle\\ &:= J_1 + J_2 + J_3.\end{aligned} \tag{3.5}$$

For J_1, (2.16) yields the estimate

$$\begin{aligned}|J_1| &\leq c\|R^{(m,n)}\|\|U^{(m)}\|^{1/2}|U^{(m)}|_{(2)}^{1/2}|AR^{(m,n)}| \\ &\leq \frac{1}{10}|AR^{(m,n)}|^2 + c|AU^{(m)}|^2\|R^{(m,n)}\|^2. \end{aligned} \tag{3.6}$$

On the other hand, (2.16) also implies

$$\begin{aligned}|J_2| &\leq C\|U^{(n)}\|\|R^{(m,n)}\|^{1/2}|R^{(m,n)}|_{(2)}^{1/2}|AR^{(m,n)}| \\ &\leq \frac{1}{10}|AR^{(m,n)}|^2 + c\|U^{(n)}\|^4\|R^{(m,n)}\|^2. \end{aligned} \tag{3.7}$$

Finally, by applying (2.9), (2.10) with $\alpha = 1/4$ and then making use of (2.17), (2.18) we estimate the third term

$$\begin{aligned}|J_3| &\leq \frac{1}{10}|AR^{(m,n)}|^2 + c|P_m^n B(U^{(n)})|^2 \\ &\leq \frac{1}{10}|AR^{(m,n)}|^2 + \frac{c}{\lambda_n^{1/4}}|B(U^{(n)})|_{1/4}^2 \\ &\leq \frac{1}{10}|AR^{(m,n)}|^2 + \frac{c}{\lambda_n^{1/4}}|B(U^{(n)})|^{3/2}\|B(U^{(n)})\|^{1/2} \\ &\leq \frac{1}{10}|AR^{(m,n)}|^2 + \frac{c}{\lambda_n^{1/4}}|B(U^{(n)})|\|B(U^{(n)})\| \\ &\leq \frac{1}{10}|AR^{(m,n)}|^2 + \frac{c}{\lambda_n^{1/4}}(\|U^{(n)}\|^{3/2}|AU^{(n)}|^{1/2})(\|U^{(n)}\|^{1/2}|AU^{(n)}|^{3/2}) \\ &\leq \frac{1}{10}|AR^{(m,n)}|^2 + \frac{c}{\lambda_n^{1/4}}\|U^{(n)}\|^2|AU^{(n)}|^2. \end{aligned} \tag{3.8}$$

At this juncture we underline that the second inequality is justified since $B(U^{(n)}) \in \tilde{V}_{1/4} = V_{1/4} = D(A^{1/8})$ (see Proposition 2.1).

For the terms involving the Coriolis operator E, we make a second application of (2.10) with $\alpha = 1/4$ and then apply (2.22) to observe:

$$\begin{aligned}&\langle P_m EU^{(m)} - P_n EU^{(n)}, AR^{(m,n)}\rangle| \\ &\quad\leq |\langle ER^{(m,n)}, AR^{(m,n)}\rangle| + |\langle P_m^n EU^{(n)}, AR^{(m,n)}\rangle| \\ &\quad\leq c(|R^{(m,n)}||AR^{(m,n)}| + |Q_n EU^{(n)}||AR^{m,n}|) \\ &\quad\leq \frac{1}{10}|AR^{(m,n)}|^2 + c(\|R^{(m,n)}\|^2 + |Q_n EU^{(n)}|^2) \\ &\quad\leq \frac{1}{10}|AR^{(m,n)}|^2 + c(\|R^{(m,n)}\|^2 + \frac{1}{\lambda_n^{1/4}}|EU^{(n)}|_{1/4}^2) \\ &\quad\leq \frac{1}{10}|AR^{(m,n)}|^2 + c(\|R^{(m,n)}\|^2 + \frac{1}{\lambda_n^{1/4}}|EU|^{3/2}\|EU^{(n)}\|^{1/2}) \\ &\quad\leq \frac{1}{10}|AR^{(m,n)}|^2 + c(\|R^{(m,n)}\|^2 + \frac{1}{\lambda_n^{(1/4)}}\|U^{(n)}\|^2) \end{aligned} \tag{3.9}$$

Along the same lines, but more involved, the term involving A_p is estimated with (2.10), (2.12)

$$
\begin{aligned}
&|\langle P_m A_p U^{(m)} - P_n A_p U^{(n)},\ AR^{(m,n)}\rangle| \\
&\quad \leq |\langle A_p R^{(m,n)}, AR^{(m,n)}\rangle| + |\langle (P_m - P_n) A_p U^{(n)}, AR^{(m,n)}\rangle| \\
&\quad \leq \frac{1}{10}|AR^{(m,n)}|^2 + c\|R^{(m,n)}\|^2 + |\langle A^{1/8} A_p U^{(n)}, (P_m - P_n) A^{7/8} R^{(m,n)}\rangle| \\
&\quad \leq \frac{1}{10}|AR^{(m,n)}|^2 + c\|R^{(m,n)}\|^2 + |A_p U^{(n)}|_{1/4} |AR^{(m,n)}| \\
&\quad \leq \frac{1}{10}|AR^{(m,n)}|^2 + c\|R^{(m,n)}\|^2 + \|A_p U^{(n)}\| \frac{1}{\lambda_n^{1/8}} |AR^{(m,n)}|) \\
&\quad \leq \frac{1}{10}|AR^{(m,n)}|^2 + c\|R^{(m,n)}\|^2 + \frac{1}{\lambda_n^{1/8}} |U^{(n)}|_{(2)} |AR^{(m,n)}| \\
&\quad \leq \frac{2}{10}|AR^{(m,n)}|AR^{(m,n)}|^2 + c\left(\|R^{(m,n)}\|^2 + \frac{1}{\lambda_n^{1/4}} |AU^{(n)}|^2 \right).
\end{aligned} \tag{3.10}
$$

Once again the second inequalities in both (3.9), (3.10) are justified since $EU^{(n)}$, $A_p U^{(n)} \in D(A^{1/4})$. Summing up the estimates above we conclude:

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|R^{(m,n)}\|^2 + |AR^{(m,n)}|^2 \\
&\quad \leq c(1 + |AU^{(m)}|^2 + \|U^{(n)}\|^4)\|R^{(m,n)}\|^2 \\
&\qquad + \frac{c}{\lambda_n^{1/4}}(1 + \|U^{(n)}\|^2)(1 + |AU^{(n)}|^2) + c|P_m^n F|^2 \\
&\quad := \phi_{m,n}\|R^{(m,n)}\|^2 + G_{m,n}.
\end{aligned} \tag{3.11}
$$

By applying (3.3) we observe that

$$
\sup_{m,n} \int_0^{t'_*} \phi_{m,n}\, \mathrm{d}s < \infty, \tag{3.12}
$$

and

$$
\lim_{m,n \to \infty} \int_0^{t'_*} G_{m,n}\, \mathrm{d}s = 0. \tag{3.13}
$$

We therefore drop the $|AR^{(m,n)}|^2$ term in (3.11) and with the Gronwall lemma, we conclude that for every $t \leq t'_*$,

$$
\begin{aligned}
\|R^{(m,n)}(t)\|^2 &\leq \exp\left(\int_0^t \phi_{m,n}\, \mathrm{d}s \right) \left(\|P_m^n U_0\|^2 + \int_0^t G_{m,n}\, \mathrm{d}s \right) \\
&\leq c\left(\|P_m^n U_0\|^2 + \int_0^{t'_*} G_{m,n}\, \mathrm{d}s \right).
\end{aligned} \tag{3.14}
$$

This proves that $U^{(n)}$ is Cauchy in $C([0,t'_*];V)$. Returning to (3.11) and integrating from 0 to t'_* yields the estimate

$$\int_0^{t'_*} |AR^{(m,n)}|^2 \leq \|P_m^n U_0\|^2 + \left(\sup_{t\in[0,t'_*]} \|R^{m,n}\|^2\right)\int_0^{t'_*} \phi_{m,n}\,\mathrm{d}s + \int_0^{t'_*} G_{m,n}\,\mathrm{d}s. \qquad (3.15)$$

We conclude that $U^{(n)}$ is Cauchy in $L^2([0,t'_*];D(A))$. The proof is complete. ■

4. Interpolation results

In this section we prove Proposition 2.1. The work consists in showing the equivalence of the spaces $V_{2\alpha}$, $\tilde{V}_{2\alpha}$, $D(A^\alpha)$ for $\alpha\in[0,1/4]$. This follows as a direct consequence of two technical results, Propositions 4.1 and 4.2 which we establish below. The inequality (2.10) is merely a restatement of the classical interpolation inequality [5, Proposition 2.3].

The first result is the following:

Proposition 4.1 *Let*

$$\mathcal{Z} := \{U=(u,v,T)\in H^1(\mathcal{M})^3 : \mathbf{v}=0 \text{ on } \Gamma_l\},$$
$$\mathcal{Z}_b := \{U=(u,v,T)\in H^1(\mathcal{M})^3 : \mathbf{v}=0 \text{ on } \Gamma_l\cup\Gamma_b\}, \qquad (4.1)$$

and define, for $s\in[0,1]$,

$$H_0^s(\mathcal{M})^3 := [H_0^1(\mathcal{M})^3, L^2(\mathcal{M})^3]_{1-s},$$
$$\mathcal{Z}_b^s := [\mathcal{Z}_b, L^2(\mathcal{M})^3]_{1-s},$$
$$\mathcal{Z}^s := [\mathcal{Z}, L^2(\mathcal{M})^3]_{1-s},$$
$$H^s(\mathcal{M})^3 := [H^1(\mathcal{M})^3, L^2(\mathcal{M})^3]_{1-s}.$$

Then,

$$H_0^s(\mathcal{M})^3 \subset \mathcal{Z}_b^s \subset \mathcal{Z}^s \subset H^s(\mathcal{M})^3, \qquad (4.2)$$

with the spaces being equipped with equivalent norms. Moreover, for all $s\in(0,1/2)$,

$$H_0^s(\mathcal{M})^3 = \mathcal{Z}_b^s = \mathcal{Z}^s = H^s(\mathcal{M})^3. \qquad (4.3)$$

Proof The proof draws on classical results in [5]. If $\partial\mathcal{M}$ were smooth, the main step, to establish (4.4), would follow precisely from [5, Theorem 11.1]. Here, due to its four corners, $\mathcal{M}$ is not smooth and further analysis is required.

The inclusions (4.2) are a direct consequence of the definitions. Let $\mathcal{D}(\mathcal{M})$ be the collection of all C^∞ functions with compact support in $\mathcal{M}$. Since,

$$\mathcal{D}(\mathcal{M})\subseteq H_0^1(\mathcal{M})\subseteq H_0^s(\mathcal{M})\subseteq H^s(\mathcal{M}),$$

the second point (4.3) follows if we can show that

$$\mathcal{D}(\mathcal{M}) \text{ is dense in } H^s(\mathcal{M}). \qquad (4.4)$$

To this end fix $u \in H^s(\mathcal{M})$. We consider a particular covering of $\mathcal{M}$ by $\mathcal{O}_0 = \mathcal{M}$, and by balls $\mathcal{O}_j$, $j \geq 1$, centred on the boundary $\partial\mathcal{M}$. We first choose four balls $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$ and $\mathcal{O}_4$ centred at the corners, each with radius small enough so as to avoid touching the opposite sides. Then, if necessary, we choose further balls all of which avoid the corners to complete the covering. Let $1 = \sum_j \phi_j$ be a partition of unity subordinated to $\{\mathcal{O}_j\}_j$. We need to show that the elements $\phi_1 u$, $\phi_2 u$, $\phi_3 u$, $\phi_4 u$, those corresponding to the corners, may be approximated by functions in $D(\mathcal{M})$. Any remaining elements $\phi_j u$ are treated exactly as in [5].

Take $\widetilde{\phi_1 u}$ to be the extension of $\phi_1 u$ by zero outside of $\mathcal{M}$. As a consequence of Lemma 4.1 established below, we see that $\widetilde{\phi_1 u}$ is in $H^s(\mathbb{R}^2)$. Consider now a smooth mollifier function ρ with support included in the ball $B := B((-1,-1), 1/4)$ centred at $(-1,-1)$ with radius $1/4$ (see e.g. [23] for basic properties). As $\epsilon \to 0$ we have the $\rho_\epsilon * v \to v$ in $H^1(\mathbb{R}^2)$ (resp. $L^2(\mathbb{R}^2)$), for every $v \in H^1(\mathbb{R}^2)$, (resp. $L^2(\mathbb{R}^2)$). Hence, by interpolation, $\rho_\epsilon * v$ converges to v in $H^s(\mathbb{R}^2)$, for every $v \in H^s(\mathbb{R}^2)$. Moreover we have that $supp(\rho_\epsilon * v)$ is included in $supp(\rho_\epsilon) + supp(v)$ [24]. Note that ρ_ϵ is supported by $B_\epsilon = B((-\epsilon,-\epsilon), \epsilon/4)$. In particular, we have $\rho_\epsilon * \widetilde{\phi_1 u}$ converges to $\widetilde{\phi_1 u}$ in $H^s(\mathbb{R}^d)$ and that (for all sufficiently small values of ϵ) $\rho_\epsilon * \widetilde{\phi_1 u}$ is a family of smooth functions, compactly supported in $\mathcal{M}$. We conclude that $\rho_\epsilon * \widetilde{\phi_1 u}|_{\mathcal{M}} \in \mathcal{D}(\mathcal{M})$ and converges to $\widetilde{\phi_1 u}|_{\mathcal{M}} = \phi_1 u$ in $H^s(\mathcal{M})$.

With similar arguments we may treat the remaining corner elements $\phi_2 u$, $\phi_3 u$, $\phi_4 u$. The proof is therefore complete. ∎

The proof of Proposition 4.1 required the following Lemma. An analogous result for the case of smooth domains was established in [5, Theorem 11.4].

LEMMA 4.1 *Suppose* $0 \leq s < 1/2$. *The extension* $\tilde{v}$ *of any element* $v \in H^s(\mathcal{M})$ *to be zero outside* $\mathcal{M}$ *defines a linear continuous operator mapping from* $H^s(\mathcal{M})$ *into* $H^s(\mathbb{R}^2)$.

Proof We proceed by making some judicious applications of certain extension and restriction operators. Given $v \in H^1(\mathcal{M})$ we define $\mathcal{E}v$ on $(-L, 0) \times (-h, 0)$ by setting

$$\mathcal{E}u(x,z) = u(-x,z), \quad -L < x < 0, -h < z < 0.$$

By similar reflections we are able to define $\mathcal{E}v$ on the strip $\mathbb{R}_x \times (-h, 0)$. Let $\mathcal{N}_1$ be any open bounded set with smooth boundary so that $\mathcal{M} \subseteq \mathcal{N}_1 \subseteq \mathbb{R}_x \times (-h, 0)$. It is clear that $\mathcal{E}$ is linear continuous from $H^1(\mathcal{M})$ to $H^1(\mathcal{N}_1)$ and from $L^2(\mathcal{M})$ to $L^2(\mathcal{N}_1)$. Hence, by interpolation, $\mathcal{E}$ is continuous from $H^s(\mathcal{M})$ to $H^s(\mathcal{N}_1)$. Next for $v \in H^s(\mathcal{N}_1)$ define $S_1 v$ to be the extension of v by zero outside of $\mathcal{N}_1$. By [5, Theorem 11.4], S_1 continuously maps $H^s(\mathcal{N}_1)$ into $H^s(\mathbb{R}_x \times \mathbb{R}_z)$. Now take $\mathcal{N}_2$ to be open bounded with smooth boundary so that $\mathcal{M} \subseteq \mathcal{N}_2 \subseteq [0, L] \times \mathbb{R}_z$. We define R to be the restriction operator to $\mathcal{N}_2$. Observe, again by interpolation, that R is a continuous map from $H^s(\mathbb{R}_x \times \mathbb{R}_z)$ into $H^s(\mathcal{N}_2)$. Finally, for $v \in H^s(\mathcal{N}_2)$ we define $S_2 v$ to be extension of v by zero outside of $\mathcal{N}_2$. Again by [5, Theorem 11.4], S_2 is linear continuous from $H^s(\mathcal{N}_2)$ into $H^s(\mathbb{R}_x \times \mathbb{R}_z)$. Summarizing we see that $S_2 R S_1 \mathcal{E}$ is linear and continuous from $H^s(\mathcal{M})$ into $H^s(\mathbb{R}^2)$. With the observation that $S_2 R S_1 \mathcal{E} v = \tilde{v}$ we conclude the desired result. ∎

We come to the second proposition which is an application of [5, Theorem 14.3]. We note that this result relies on holomorphic interpolation which, for our context, gives the same spaces as Hilbert interpolation [5, Theorem 14.1, Remark 14.2].

PROPOSITION 4.2 *For all* $s \in [0, 1]$,

$$V_s = \mathcal{Z}_b^s \cap H, \quad \tilde{V}_s = \mathcal{Z}^s \cap H. \tag{4.5}$$

Remark 4.1 We may endow both $\mathcal{Z}$ and $\mathcal{Z}_b$ with the $\|\cdot\|$ norm defined in Section 2. As noted above Π is continuous on $H^1(\mathcal{M})^3 \supset \mathcal{Z}, \mathcal{Z}_b, H_0^1(\mathcal{M})$. This observation is used in the proof of Lemma 4.2.

Proof In this proof we follow closely the notations used in [5]. We begin by addressing V_s. We define

$$X = \mathcal{Z}_b, \quad Y = \Phi = L^2(\mathcal{M})^3, \tag{4.6}$$

and observe that these spaces satisfy (14.18) of [5]. We take $\partial := I - \Pi$, recalling that Π is the projection operator from $L^2(\mathcal{M})$ onto H (see Section 2). Let

$$\mathfrak{X} = \{0\} = \mathfrak{Y}, \quad \Psi = L^2(\mathcal{M})^d.$$

Trivially, $\mathfrak{Y}, \mathfrak{X} \subset \Psi$, are each Banach and $\partial \in \mathcal{L}(\Phi, \Psi)$ which are the requirements of (14.19), (14.20) in [5]. Following the notations set in (14.21), (14.22) of [5] we observe that

$$(X)_{\partial,\mathfrak{X}} = \{U \in \mathcal{Z}_b : \partial U = 0\} = \{U \in \mathcal{Z}_b : \Pi U = U\} = \mathcal{Z}_b \cap H = V, \tag{4.7}$$

and

$$(Y)_{\partial,\mathfrak{Y}} = \{U \in L^2(\mathcal{M})^3 : \Pi U = U\} = H. \tag{4.8}$$

It remains to address (14.23), the remaining condition necessary to apply [5, Theorem 14.3]. We take

$$\tilde{\mathfrak{X}} := \partial \mathcal{Z}_b \quad \tilde{\mathfrak{Y}} := \partial L^2(\mathcal{M})^d = H^\perp \tag{4.9}$$

and observe that each space is Banach. This is clear for $\tilde{\mathfrak{Y}}$ and for the space $\tilde{\mathfrak{X}}$ we make use of the fact that ∂ is a projection operator, i.e. that $\partial^2 U = \partial U$ over all $U \in L^2(\mathcal{M})^3$. With this in mind, suppose that $\{U_n\}_{n \geq 0}$ is a convergent sequence in $\partial \mathcal{Z}_b$ and denote the limit by U. Clearly there must exist a sequence $\{U_n^\sharp\}_{n \geq 0}$ in $\mathcal{Z}_b$ such that $\partial U_n^\sharp \to U$ (with the convergence understood in the topology defined by $\|\cdot\|$). Since $\partial = I - \Pi$ is continuous on $H^1(\mathcal{M})^3$, we infer that $\partial^2 U_n^\sharp \to \partial U$. Thus, since ∂ is a projection this in turn implies $\partial U_n^\sharp \to \partial U$ so that $\partial U = U$. We thus infer that $\partial \mathcal{Z}_b$ is equal to its closure and hence is Banach. Since ∂ is continuous both in $L^2(\mathcal{M})^3$ and $H^1(\mathcal{M})^3$ it follows directly that $\partial \in \mathcal{L}(X, \tilde{\mathfrak{X}}) \cap \mathcal{L}(Y, \tilde{\mathfrak{Y}})$. This is the requirement of (14.23), (ii). For (14.23), (iii) we take $\mathcal{G} := \partial$ and $r := 0$. With these definitions one may see by inspection that the identity $\partial \mathcal{G} \chi = \chi + r\chi$, holds for every $\chi \in \tilde{\mathfrak{X}} + \tilde{\mathfrak{Y}}$.

Gathering the above observations we now infer from Theorem 14.3 that

$$\begin{aligned} V_s &= [V, H]_{1-s} = [(X)_{\partial,\mathfrak{X}}, (Y)_{\partial,\mathfrak{X}}]_{1-s} \\ &= ([X, Y]_{1-s})_{\partial,[\mathfrak{X},\mathfrak{Y}]_{1-s}} = ([\mathcal{Z}_b, L^2(\mathcal{M})^3]_{1-s})_{\partial,\mathfrak{X}} = (\mathcal{Z}_b^s)_{\partial,\mathfrak{X}} \\ &= \{U \in \mathcal{Z}_b^s : \Pi U = U\} = \mathcal{Z}_b^s \cap H. \end{aligned} \tag{4.10}$$

To address $\tilde{V}_s$ the proof follows exactly as above by replacing $\mathcal{Z}$ by $\mathcal{Z}_b$ in (4.6), (4.7), (4.9), (4.10) and by replacing $\tilde{V}$ by V in (4.7), (4.10). ■

5. Cauchy convergence for the Navier-Stokes equations

We finally consider the Navier-Stokes equations in space dimension $d=2$, 3, on a bounded domain $\mathcal{M}$:

$$\partial_t U + (U \cdot \nabla)U - \nu \Delta U + \nabla p = F, \tag{5.1a}$$

$$\nabla \cdot U = 0, \tag{5.1b}$$

$$U(0) = U_0, \tag{5.1c}$$

$$U_{|\mathcal{M}} = 0. \tag{5.1d}$$

The system (5.1) describes the flow of a viscous incompressible fluid. Here $U=(u_1, \dots, u_d)$, p and ν represent the velocity field, the pressure and the coefficient of kinematic viscosity, respectively. We assume that $\mathcal{M}$ has a smooth boundary $\partial\mathcal{M}$.

We begin by recalling the abstract setting for (5.1). See e.g. [22] for a detailed treatment. Let $H := \{U \in L^2(\mathcal{M})^d : \nabla \cdot U = 0, U \cdot n = 0\}$, where n is the outer pointing normal to $\partial\mathcal{M}$; H is endowed as a Hilbert space with the L^2 inner product and norm. The Leray–Hopf projector, P_H, is defined as the orthogonal projection of $L^2(\mathcal{M})^d$ onto H. Also define $V := \{U \in H_0^1(\mathcal{M})^d : \nabla \cdot U = 0\}$, with the inner product $((U, U^\sharp)) = \int_{\mathcal{M}} \nabla U \cdot \nabla U^\sharp \, d\mathcal{M}$. Due to the Dirichlet boundary condition, (5.1d), the Poincaré inequality $|U| \leq c\|U\|$ holds for $U \in V$ justifying this defintion.

The linear portion of (5.1) is captured in the Stokes operator $A = -P_H \Delta$, which is an unbounded operator from H to H with the domain $D(A) = H^2(\mathcal{M}) \cap V$. As above, one can prove the existence of an orthonormal basis $\{\Phi_k\}_{k\geq 0}$ for H of eigenfunctions of A with the associated eigenvalues $\{\lambda_k\}_{k\geq 0}$ forming an unbounded increasing sequence. Define $H_n = span\{\Phi_1, \dots, \Phi_n\}$ and take P_n to be the projection from H onto this space. We let $Q_n := I - P_n$ and $P_n^m := P_m - P_n$ for every mn.

As above (cf (2.6), (2.7), (2.8)) we may define the fractional powers A^α. We denote the domains of these operators by $D(A^\alpha)$ and associate norms $|\cdot|_{2\alpha} = |A^\alpha \cdot|$. With these definitions and notations the generalized direct and inverse Poincaré estimates hold as in (2.9). Also in parallel to the presentation above we let $\tilde{V} = H \cap H^1(\mathcal{M})^d$ and define the intermediate spaces $\tilde{V}_s = [\tilde{V}, H]_{1-s}$, $V_s = [V, H]_{1-s}$. As previously in Lemma 2.1 we see that

$$V_{2\alpha} = \tilde{V}_{2\alpha} = D(A^\alpha) \quad \text{for all } \alpha \in (0, 1/4) \tag{5.2}$$

and that

$$|U|_\alpha \leq |U|^{1-\alpha} \|U\|^\alpha_{H^1(\mathcal{M})} \quad \text{whenever } U \in \tilde{V}. \tag{5.3}$$

Indeed, due to [5, Theorem 11.1] we have that

$$H_0^s(\mathcal{M})^d = H^s(\mathcal{M})^d$$

for all $s \in [0, 1/2]$. As such (5.2) and (5.3) follow directly once we establish that, for $s \in (0, 1)$

$$\tilde{V}_s = H^s(\mathcal{M}) \cap H, \quad V_s = H_0^s(\mathcal{M}) \cap H. \tag{5.4}$$

Perusing the proof of Lemma 4.2 we see that (5.4) is established in exactly the same manner modulo with some minor notational changes.

The nonlinear portion of (5.1) is given by $B(U, U^\sharp) := P_H((U \cdot \nabla)U^\sharp) = P_H(\sum_{j=1}^d u_j \partial_j U^\sharp)$, which is defined for $U = (u_1, \ldots, u_d) \in V$ and $U^\sharp \in D(A)$. We use the following properties of B.

LEMMA 5.1 *Suppose $d=2$ or 3.*

(i) *B is bilinear and continuous from $V \times D(A)$ to H. If $U \in V$, $U^\sharp \in D(A)$, and $U^\flat \in H$, then*

$$\left|(B(U, U^\sharp), U^\flat)\right| \leq c \begin{cases} |U|^{1/2}\|U\|^{1/2}\|U^\sharp\|^{1/2}|AU^\sharp|^{1/2}|U^\flat| & \text{in } d=2, \\ \|U\|\|U^\sharp\|^{1/2}|AU^\sharp|^{1/2}|U^\flat| & \text{in } d=3. \end{cases} \tag{5.5}$$

(ii) *If $U \in D(A)$, then $B(U) \in \tilde{V}$, and for $d=2,3$ we have,*

$$\|B(U)\|^2_{H^1(\mathcal{M})} \leq c\|U\||AU|^3 + |U|^{1/2}|AU|^{7/2}, \tag{5.6}$$

and, for every $s \in (0, 1/2)$,

$$\begin{aligned} |B(U)|^2_{\tilde{V}_s} &\leq |B(U)|^{2(1-s)}\|B(U)\|^{2s}_{H^1(\mathcal{M})^d} \\ &\leq c(\|U\|^{3-2s}|AU|^{1+2s} + |U|^{s/2}\|U\|^{3-3s}|AU|^{1+(5/2)s}). \end{aligned} \tag{5.7}$$

Remark 5.1 It is incorrectly stated in [4] Lemma 2.2 (iii) that $B(U) \in V$ when $u \in D(A)$. Indeed $B(U) = (U \cdot \nabla)U - \nabla q$ where $q = q(U)$ is the unique element in $H^1(\mathcal{M})$ such that $(U \cdot \nabla)U - \nabla q \in H$. Hence ∇q and thus $B(U)$ do not necessarily vanish on $\partial\mathcal{M}$, so that $B(U)$ is not necessarily in V. This oversight invalidates the estimate (3.17) upon which (3.18) and thus the conclusion of [4, Proposition 3.1(i)] rely.

Fortunately the estimate in [4, Lemma 2.2] can be replaced by (5.6) given here. Indeed the proof of [4, Lemma 2.2] gives valid estimates similar to (5.6) for $(U \cdot \nabla)U$ in $H^1(\mathcal{M})^d$. We know moreover that the Leray–Hopf projection of $L^2(\mathcal{M})^d$ onto H is continuous in $H^1(\mathcal{M})^d$ [22]. We may thus infer that $B(U)$ is also in $H^1(\mathcal{M})^d$ and satisfies the same estimates, proving (5.6). Using (5.2) and (5.3), (5.7) follows immediately from (5.6)

In Theorem 5.1 we establish (5.12) which may be taken as a replacement of (3.17) in [4, Proposition 3.1(i)]. With minor changes in (3.18) this fixes the oversight in this previous work.

With the definitions for A and B and assuming that $F \in L^2_{loc}([0, \infty); H)$ and that $U_0 \in V$ we formulate (5.1) as the abstract evolution on H:

$$\frac{dU}{dt} + AU + B(U) = F \quad U(0) = U_0. \tag{5.8}$$

As above we define the associated Galerkin approximations as the solutions $U^n \in C([0, \infty); H_n)$ of

$$\frac{dU^{(n)}}{dt} + AU^{(n)} + P_n B(U^{(n)}) = P_n F \quad U^{(n)}(0) = P_n U_0. \tag{5.9}$$

Similarly to Section 3 we have the Cauchy convergence of $\{U^{(n)}\}_{n \geq 1.}$

THEOREM 5.1 *Suppose $d=2,3$ and that $F \in L^2_{loc}([0,\infty); H)$, $U_0 \in V$. Then there exists $t'_* > 0$ such that $\{U^{(n)}\}_{n\geq 1}$ is Cauchy in $C([0,t'_*]; V) \cap L^2([0,t'_*]; D(A))$. In the case $d=2$ the result is global; more precisely $\{U^{(n)}\}_{n\geq 1}$ is Cauchy in $C([0,t'_*]; V) \cap L^2([0,t'_*]; D(A))$ for every $t'_* > 0$.*

Proof Multiplying (5.9) by $AU^{(n)}$, integrating and applying (5.5) leads to the estimates,

$$\frac{d}{dt}\|U^{(n)}\|^2 + |AU^{(n)}|^2 \leq c\begin{cases} 1 + |P_nF|^2 + |U^{(n)}|^2\|U^{(n)}\|^4 & d=2, \\ 1 + |P_nF|^2 + \|U^{(n)}\|^6 & d=3. \end{cases} \tag{5.10}$$

From these inequalities we infer the existence of $t'_* > 0$ which satisfies uniform bounds as in (3.3). Note however that for $d=2$ we can take t'_* arbitrarily large.

Defining $R^{(m,n)} = U^{(m)} - U^{(n)}$ for all pairs of $m > n$ the proof proceeds similarly to Theorem 3.1. The only significant difference comes when we estimate J_3 coming from the analogue of (3.5). Consider any $\alpha \in [0, 1/5)$. Due to (5.2), $D(A^\alpha) = V_{2\alpha} = \tilde{V}_{2\alpha}$. Using (5.7) with $s = 2\alpha$, we estimate

$$\begin{aligned} T_3 &\leq \frac{\nu}{12}|AR^{(m,n)}|^2 + c|P_n^m B(U^{(n)})|^2 \\ &\leq \frac{\nu}{12}|AR^{(m,n)}|^2 + \frac{c}{\lambda_n^{2\alpha}}|Q_n B(U^{(n)})|^2_{2\alpha} \\ &\leq \frac{\nu}{12}|AR^{(m,n)}|^2 + \frac{c}{\lambda_n^{2\alpha}}|B(U^{(n)})|^2_{V_{2\alpha}} \\ &\leq \frac{\nu}{12}|AR^{(m,n)}|^2 \\ &\quad + \frac{c}{\lambda_n^{2\alpha}}(\|U^{(n)}\|^{3-4\alpha}|AU^{(n)}|^{1+4\alpha} + |U^{(n)}|^\alpha\|U^{(n)}\|^{3-6\alpha}|AU^{(n)}|^{1+5\alpha}) \\ &\leq \frac{\nu}{12}|AR^{(m,n)}|^2 \\ &\quad + \frac{c}{\lambda_n^{2\alpha}}(\|U^{(n)}\|^{\frac{2(3-4\alpha)}{1-4\alpha}} + |U^{(n)}|^{\frac{2\alpha}{1-5\alpha}}\|U^{(n)}\|^{\frac{2(3-6\alpha)}{1-5\alpha}} + |AU^{(n)}|^2). \end{aligned} \tag{5.11}$$

Note that these bounds are valid for both $d=2,3$. For definiteness we take $\alpha = 1/8$ and replace (3.8) with

$$T_3 \leq \frac{\nu}{12}|AR^{(m,n)}|^2 + \frac{c}{\lambda_n^{1/4}}(\|U^{(n)}\|^{10} + |U^{(n)}|^{2/3}\|U^{(n)}\|^{12} + |AU^{(n)}|^2). \tag{5.12}$$

With inconsequential changes to the definitions of $\phi_{m,n}$, $G_{m,n}$ in (3.11) we conclude the desired results. ■

Acknowledgements

This work was partially supported by the National Science Foundation under the grants NSF-DMS-0604235 and NSF-DMS-0906440, and by the Research Fund of Indiana University.

Notes

1. One sometimes also finds the more general definition $(U, U^\sharp) := \int_{\mathcal{M}} \mathbf{v} \cdot \mathbf{v}^\sharp \, d\mathcal{M} + \kappa \int_{\mathcal{M}} TT^\sharp \, d\mathcal{M}$ with $\kappa > 0$ fixed. This κ is useful for the coherence of physical dimensions and for (mathematical) coercivity. Since this is not needed here we take $\kappa = 1$.

2. B is also continuous from $V \times V$ to V' and satisfies important cancellation properties and estimates. Since we are considering strong solutions in $C([0,t],V) \cap L^2([0,t],D(A))$ these properties will not be needed here.

References

[1] J.P. Aubin, *Un théorème de compacité*, C. R. Acad. Sci. Paris 256 (1963), pp. 5042–5044.

[2] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod/Gauthier-Villars, Paris, 1969, xx + 554 pp. (in French).

[3] N. Glatt-Holtz and R. Temam, Pathwise solutions of the 2-d stochastic primitive equations, Applied Mathematics & Optimization (accepted).

[4] N. Glatt-Holtz and M. Ziane, *Strong pathwise solutions of the stochastic Navier-Stokes system*, Adv. Differ. Equ. 14(5–6) (2009), pp. 567–600.

[5] J.-L. Lions E. Magenes, *Non-homogeneous Boundary Value Problems and Applications*, Die Grundlehren der Mathematischen Wissenschaften, Band 181, Vol. I, Springer-Verlag, New York, 1972 (Translated from French by P. Kenneth).

[6] K. Trènberth, *Climate System Modeling*, 1st ed., Cambridge University Press, Cambridge, 1993.

[7] J.-L. Lions, R. Temam, and S.H. Wang, *On the equations of the large-scale ocean*, Nonlinearity 5(5) (1992), pp. 1007–1053.

[8] J.-L. Lions, R. Temam, and S.H. Wang, *New formulations of the primitive equations of atmosphere and applications*, Nonlinearity 5(2) (1992), pp. 237–288.

[9] J.-L. Lions, R. Temam, and S. Wang, *Models for the coupled atmosphere and ocean. (CAO I,II)*, Comput. Mech. Adv. 1(1) (1993), p. 120.

[10] C. Cao and E. Titi, *Global well-posedness of the three-dimensional viscous primitive equations of large scale ocean and atmosphere dynamics*, Ann. Math. (2) 166(1) (2007), pp. 245–267.

[11] G.M. Kobelkov, *Existence of a solution 'in the large' for the 3D large-scale ocean dynamics equations*, C. R. Math. Acad. Sci. Paris 343(4) (2006), pp. 283–286.

[12] I. Kukavica and M. Ziane, *On the regularity of the primitive equations of the ocean*, Nonlinearity 20(12) (2007), pp. 2739–2753.

[13] M. Petcu, R. Temam, and D. Wirosoetisno, *Existence and regularity results for the primitive equations in two space dimensions*, Commun. Pure Appl. Anal. 3(1) (2004), pp. 115–131.

[14] N. Glatt-Holtz and M. Ziane, *The stochastic primitive equations in two space dimensions with multiplicative noise*, Discrete Contin. Dyn. Syst. Ser. B 10(4) (2008), pp. 801–822.

[15] B. Ewald, M. Petcu, and R. Temam, *Stochastic solutions of the two-dimensional primitive equations of the ocean and atmosphere with an additive noise*, Anal. Appl. (Singap.) 5(2) (2007), pp. 183–198.

[16] M. Petcu, R. Temam, and M. Ziane, *Some Mathematical problems in geophysical fluid dynamics*, in in *Special Volume on Computational Methods for the Atmosphere and the Oceans,* Handbook of Numerical Analysis, Vol. 14, R. Temam and J. Tribbia, eds., Elsevier, Oxford, 2008, pp. 577–750.

[17] R. Temam, A. Rousseau, and J. Tribbia, *Boundary value problems for the inviscid primitive equations in limited domain*, in in *Special Volume on Computational Methods for the Atmosphere and the Oceans*, Handbook of Numerical Analysis, Vol. 14, R. Temam and J. Tribbia, eds., Elsevier, Oxford, 2008, pp. 577–750.

[18] B. Cushman-Roisin and J.-M. Beckers, *Introduction to Geophysical Fluid Dynamics: Physical and Numerical Aspects*, Academic Press, North-Holland, Amsterdam, 2010.

[19] J. Pedlosky, *Geophysical Fluid Dynamics*, Springer Verlag, New York, 1982.

[20] M. Ziane, *Regularity results for Stokes type systems*, Appl. Anal. 58(3–4) (1995), pp. 263–292.
[21] R. Temam and M. Ziane, *Some Mathematical Problems in Geophysical Fluid Dynamics*, in *Handbook of Mathematical Fluid Dynamics*, Vol. III, North-Holland, Amsterdam, 2004, pp. 535–657.
[22] R. Temam, *Navier–Stokes Equations: Theory and Numerical Analysis*, AMS Chelsea Publishing, Providence, RI, 2001 (Reprint of the 1984 edition).
[23] G.B. Folland, *Real Analysis*: *Pure and Applied Mathematics*, 2nd ed., Modern Techniques and their Applications, John Wiley & Sons, New York, 1999 (A Wiley-Interscience Publication).
[24] L. Schwartz, *Théorie des Distributions*, Tomes i, ii, Hermann & Cie. Paris, 1950–1951. (Actualités Sci. Ind., no. 1091, Publ. Inst. Math. Univ. Strasbourg 9) (in French).