MATH 301/601, Solutions to test 1

1. Statements (a), (b), (d), (e), and (f) are always true; (b) is only true for independent events and in (g), there might sometimes be equality but only by coincidence.

2(a) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 1/2 + 1/2 - 1/2 \cdot 1/2 = 3/4$
(b) $P(A \cup B) = P(A) + P(B) = 1$

3. Denote the classrooms by C_1 and C_2 and let F be the event to get a female student. By LTP,

$$P(F) = P(F|C_1)P(C_1) + P(F|C_2)P(C_2) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{\binom{10}{2}}{\binom{20}{2}} = \frac{47}{76} \approx 0.62$$

4. Introduce the events D for having the disease, H for being healthy, and $+$ for testing positive. By Bayes' rule

$$P(D|+) = \frac{P(+|D)P(D)}{P(+|D)P(D) + P(+|H)P(H)} = \frac{1 \cdot 0.0001}{1 \cdot 0.0001 + 0.00 \cdot 0.9999} = 0.0099$$

5. The total number of hands is $\binom{52}{13}$. You can get 6 hearts is $\binom{13}{6}$ ways and 3 diamonds in $\binom{13}{3}$ ways. There are $\binom{13}{4}$ ways to get the remaining cards suited in spades and $\binom{13}{4}$ ways to get them suited in clubs and we get the probability

$$\frac{\binom{13}{6}\binom{13}{3}\binom{13}{4} \cdot 2}{\binom{52}{13}}$$

With the alternative interpretation that the remaining cards can be suited in any suit, $\binom{13}{4} \cdot 2$ should be replaced by $\binom{13}{4} \cdot 2 + \binom{7}{4} + \binom{10}{4}$.

6. Each cell dies with probability 1/3 and divides with probability 2/3. If it divides, one daughter cell dies with probability /12 so the probability of one surviving daughter cell is $2/3 \cdot 1/2 = 1/3$. Similarly, the probability of two surviving daughter cells is 1/3. Denote the event of extinction by E, let $q = P(E)$, and let A_k be the event that the first cell has k daughter cells. If there are no daughter cells, extinction is certain. If there is one daughter cell, extinction has probability q (population starts over from one individual) and if there are two daughter cells, extinction has probability q^2 (two independent populations must go extinct). LTP gives

$$q = P(E) = P(E|A_0)P(A_0) + P(E|A_1)P(A_1) + P(E|A_2)P(A_2)$$

$$= 1 \cdot 1/3 + q \cdot 1/3 + q^2 \cdot 1/3$$

which gives the equation $q^2 - 2q + 1 = 0$ with the only solution $q = 1$.