# MACMAHON'S SUMS-OF-DIVISORS AND ALLIED $q$-SERIES

TEWODROS AMDEBERHAN, KEN ONO AND AJIT SINGH

Abstract. Here we investigate the $q$-series

$$\mathcal{U}_a(q) = \sum_{n=0}^{\infty} MO(a;n)q^n := \sum_{0<k_1<k_2<\cdots<k_a} \frac{q^{k_1+k_2+\cdots+k_a}}{(1-q^{k_1})^2(1-q^{k_2})^2\cdots(1-q^{k_a})^2},$$

$$\mathcal{U}_a^{\star}(q) = \sum_{n=0}^{\infty} M(a;n)q^n := \sum_{1\leq k_1\leq k_2\leq\cdots\leq k_a} \frac{q^{k_1+k_2+\cdots+k_a}}{(1-q^{k_1})^2(1-q^{k_2})^2\cdots(1-q^{k_a})^2}.$$

MacMahon introduced the $\mathcal{U}_a(q)$ in his seminal work on partitions and divisor functions. Recent works show that these series are sums of quasimodular forms with weights $\leq 2a$. We make this explicit by describing them in terms of Eisenstein series. We use these formulas to obtain explicit and general congruences for the coefficients $MO(a;n)$ and $M(a;n)$. Notably, we prove the conjecture of Amdeberhan-Andrews-Tauraso as the $m=0$ special case of the infinite family of congruences

$$MO(11m+10; 11n+7) \equiv 0 \pmod{11},$$

and we prove that

$$MO(17m+16; 17n+15) \equiv 0 \pmod{17}.$$

We obtain further formulae using the limiting behavior of these series. For $n \leq a + \binom{a+1}{2}$, we obtain a "hook length" formulae for $MO(a;n)$, and for $n \leq 2a$, we find that $M(a;n) = \binom{a+n-1}{n-a} + \binom{a+n-2}{n-a-1}$.

## 1. Introduction and Statement of Results

At first glance, one might underestimate the value of the trivial observation that the number of partitions of an integer $n$ into identical parts is also the number of divisors of $n$. This fact is a glimpse of a rich theory that relates integer partitions and divisor functions. Indeed, MacMahon's important paper [7] is based on the idea of connecting partitions to divisor sums: partition of $n$ using $k_1$ repeated $s_1$ times, and $k_2$ repeated $s_2$ times, and so on through $k_a$ repeated $s_a$ times. Using this convention, he considered the sum of products of the *multiplicities* $MO(a;n) := \sum s_1 s_2 \cdots s_a$ of size $n$ partitions, which has the generating function

$$\mathcal{U}_a(q) := \sum_{n\geq 0} MO(a;n)\, q^n = \sum_{0<k_1<k_2<\cdots<k_a} \frac{q^{k_1+k_2+\cdots+k_a}}{(1-q^{k_1})^2(1-q^{k_2})^2\cdots(1-q^{k_a})^2}. \tag{1.1}$$

His work [7] is populated with beautiful divisor function identities, where $\sigma_\nu(n) := \sum_{d|n} d^\nu$, such as:

$$\mathcal{U}_1(q) = \sum_{n\geq 1} \sigma_1(n) q^n \quad \text{and} \quad \mathcal{U}_2(q) = \sum_{n\geq 1} \left( \frac{\sigma_1(n)}{8} - \frac{n\sigma_1(n)}{4} + \frac{\sigma_3(n)}{8} \right) q^n. \tag{1.2}$$

---

*Date*: November 13, 2023.
*Key words and phrases.* divisor functions, partitions, quasimodular forms.
2020 *Mathematics Subject Classification.* 11F03, 11A25, 11F50, 11M41, 11F11, 11F33.

To entice the reader, we offer the first few terms of $\mathcal{U}_1(q), \ldots, \mathcal{U}_4(q)$:

$$\mathcal{U}_1(q) = q + 3q^2 + 4q^3 + 7q^4 + 6q^5 + 12q^6 + 8q^7 + \ldots,$$
$$\mathcal{U}_2(q) = q^3 + 3q^4 + 9q^5 + 15q^6 + 30q^7 + 45q^8 + 67q^9 + \ldots,$$
$$\mathcal{U}_3(q) = q^6 + 3q^7 + 9q^8 + 22q^9 + 42q^{10} + 81q^{11} + 140q^{12} + \ldots,$$
$$\mathcal{U}_4(q) = q^{10} + 3q^{11} + 9q^{12} + 22q^{13} + 51q^{14} + 97q^{15} + 188q^{16} + \ldots.$$

The inequalities in definition (1.1) imply that $q^{-\frac{a(a+1)}{2}} \cdot \mathcal{U}_a(q) = 1 + 3q + \ldots$, while for $a \geq 2$, we have

$$q^{-\frac{a(a+1)}{2}} \cdot \mathcal{U}_a(q) = 1 + 3q + 9q^2 + \ldots.$$

Answering the natural question, we show that this sequence converges to a simple infinite product, which, by the theory of Nekrasov-Okounkov [8], gives *hook length* formulae for many of the $MO(a;n)$.

To make this precise, recall that a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of $n$, denoted $\lambda \vdash n$, is a non-increasing sequence of positive integers that sum to $n$. Its *Young diagram* is the left-justified array of boxes where the row lengths are the parts. The *hook* $H(i,j)$ of the box in position $(i,j)$ consists of this box, together with those below it and those to its right. Its *hook length* $h(i,j) := (\lambda_i - i) + (\lambda'_j - j) + 1$ is the number of such boxes, where $\lambda'_j$ is the number of boxes in column $j$. Denote the multiset of hook lengths of $\lambda$ by $\mathcal{H}(\lambda)$. Finally, we recall the "exponential form" of a partition $\lambda = (1^{m_1}, 2^{m_2}, \ldots, t^{m_t})$, where $m_i$ is the multiplicity of part $i$.

*Example.* The exponential form of $\lambda = (4,4,2)$ is $\lambda = (1^0, 2^1, 3^0, 4^2, 5^0, 6^0, 7^0, 8^0, 9^0, 10^0) \vdash 10$. Its Young diagram is given below, and shows that $\mathcal{H}(\lambda) = \{6,5,5,4,3,2,2,2,1,1\}$.

| | | | |
|---|---|---|---|
| 6 | 5 | 3 | 2 |
| 5 | 4 | 2 | 1 |
| 2 | 1 | | |

We derive the following result using the work of Andrews-Rose [2] and Nekrasov-Okounkov [8].

**Theorem 1.1.** *The following are true:*
*(i) If $a$ is a positive integer, then we have that*

$$q^{-\frac{a(a+1)}{2}} \cdot \mathcal{U}_a(q) = \prod_{n\geq 1} \frac{1}{(1-q^n)^3} + O(q^{a+1}).$$

*(ii) If $n \leq a + \binom{a+1}{2}$, then we have that*

$$MO(a;n) = \sum_{\lambda \vdash n-a} \prod_{h \in \mathcal{H}(\lambda)} \left(\frac{2}{h^2} + 1\right) = \sum_{\lambda \vdash n-a} \prod_{s=1}^{n-a} \binom{2+m_s}{2}.$$

Inspired by the $\mathcal{U}_a(q)$, Amdeberhan-Andrews-Tauraso [1] initiated the study of the $q$-series

$$\mathcal{U}_a^{\star}(q) := \sum_{n\geq 0} M(a;n)\, q^n = \sum_{1\leq k_1 \leq k_2 \leq \cdots \leq k_a} \frac{q^{k_1+k_2+\cdots+k_a}}{(1-q^{k_1})^2(1-q^{k_2})^2\cdots(1-q^{k_a})^2}, \tag{1.3}$$

where the strict inequalities in (1.1) are replaced by weak inequalities. One easily sees that

$$\mathcal{U}_a^{*}(q) = \sum_{n\geq 0} M(a;n) q^n = q^a + (2a+1)q^{a+1} + \ldots.$$

To entice the reader, we offer the first few terms of $\mathcal{U}_1^*(q), \ldots, \mathcal{U}_4^*(q)$:

$$\mathcal{U}_1^\star(q) = q + 3q^2 + 4q^3 + 7q^4 + 6q^5 + 12q^6 + \cdots ,$$
$$\mathcal{U}_2^\star(q) = q^2 + 5q^3 + 14q^4 + 29q^5 + 55q^6 + 86q^7 + \cdots ,$$
$$\mathcal{U}_3^\star(q) = q^3 + 7q^4 + 27q^5 + 77q^6 + 181q^7 + 378q^8 + \cdots ,$$
$$\mathcal{U}_4^\star(q) = q^4 + 9q^5 + 44q^6 + 156q^7 + 450q^8 + 1121q^9 + \cdots .$$

In analogy with Theorem 1.1, we consider the limiting behavior of these series. These series converge to specializations of the generating function for the polynomials $p_0(x) := 1, p_1(x) := 2x+1, p_2(x) := 2x^2+3x, p_3(x) := \frac{4}{3}x^3 + 4x^2 + \frac{5}{3}x, \ldots$. For $n \geq 1$, these polynomials are defined by

$$p_n(x) := \binom{2x+n-1}{n} + \binom{2x+n-2}{n-1}. \tag{1.4}$$

As a companion to Theorem 1.1, we obtain the following theorem.

**Theorem 1.2.** *The following are true:*
*(i) If $a$ is a positive integer, then we have that*

$$q^{-a} \cdot \mathcal{U}_a^*(q) = \sum_{n=0}^{a} p_n(a) q^n + O(q^{a+1}).$$

*(ii) If $n \leq 2a$, then we have that $M(a;n) = p_{n-a}(a)$.*

*Remark.* The $\mathcal{U}_a(q)$ and $\mathcal{U}_a^\star(q)$ are multiple $q$-zeta values. To make this precise, we recall the $q$-notation $[k]_q := \frac{1-q^k}{1-q}$ and the multiple $q$-zeta values (for example, see [3])

$$\zeta_q(m_1, \ldots, m_a) := \sum_{0<k_1<\cdots<k_a} \frac{q^{(m_1-1)k_1+\cdots+(m_a-1)k_a}}{[k_1]_q^{m_1} \cdots [k_a]_q^{m_a}},$$
$$\zeta_q^\star(m_1, \ldots, m_a) := \sum_{1\leq k_1\leq\cdots\leq k_a} \frac{q^{(m_1-1)k_1+\cdots+(m_a-1)k_a}}{[k_1]_q^{m_1} \cdots [k_a]_q^{m_a}}.$$

We have that $(1-q)^{2a} \cdot \mathcal{U}_a(q) = \zeta_q(2, \ldots, 2)$ and $(1-q)^{2a} \cdot \mathcal{U}_a^\star(q) = \zeta_q^\star(2, \ldots, 2)$.

As divisor functions arise as the coefficients of Eisenstein series, identities such as (1.2) suggest a strong relationship between the $\mathcal{U}_a(q)$ and quasimodular forms. This speculation was confirmed by Andrews-Rose. Indeed, they proved (see [2, Cor. 4]) and [12, Th. 1.12]) that each $\mathcal{U}_a(q)$ is a linear combination of quasimodular forms on $\mathrm{SL}_2(\mathbb{Z})$ with weights $\leq 2a$. Similarly, Amdeberhan-Andrews-Tauraso [1, Th. 6.1] proved that each $\mathcal{U}_a^\star(q)$ is a linear combination of quasimodular forms on $\mathrm{SL}_2(\mathbb{Z})$ with weights $\leq 2a$.

Here we make this quasimodularity explicit. In the case of $\mathcal{U}_a(q)$, we employ the standard generators of the graded ring of quasimodular forms: the quasimodular weight 2 Eisenstein series

$$E_2(q) := 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n, \tag{1.5}$$

and the weight 4 and 6 modular Eisenstein series

$$E_4(q) := 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n \qquad \text{and} \qquad E_6(q) := 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n. \tag{1.6}$$

It is well known [6] that the ring of quasimodular forms is $\mathbb{C}[E_2, E_4, E_6]$, and so our goal is to obtain formulas in terms of the monomials $E_2^{\alpha}(q)E_4^{\beta}(q)E_6^{\gamma}(q)$, where $\alpha, \beta$ and $\gamma$ are non-negative integers.

Our formulas for $\mathcal{U}_a(q)$ use of the triple index sequence of rational numbers defined by the recursion

$$\begin{aligned} c(\alpha,\beta,\gamma) := -\frac{1}{3}(2\alpha+8\beta+12\gamma+1)\cdot c(\alpha-1,\beta,\gamma)+\frac{2}{3}(\alpha+1)\cdot c(\alpha+1,\beta-1,\gamma) \\ +\frac{8}{3}(\beta+1)\cdot c(\alpha,\beta+1,\gamma-1)+4(\gamma+1)\cdot c(\alpha,\beta-2,\gamma+1), \end{aligned} \tag{1.7}$$

where $\alpha, \beta, \gamma \geq 0$. To seed the recursion, we let $c(0,0,0) := 1$, and we let $c(\alpha,\beta,\gamma) := 0$ if any of the arguments are negative. Here we list the "first few" values:

$$c(1,0,0) = -1,\ c(0,1,0) = -\frac{2}{3},\ c(0,0,1) = -\frac{16}{9},\ c(1,1,0) = \frac{14}{3},\ c(1,0,1) = \frac{64}{3}, \ldots.$$

We also require constants for the quasimodular summands sorted by weight. For $0 \leq t \leq a$, define

$$w_t(a) := \frac{\binom{2a}{a}}{16^a(2a+1)} \sum_{0\leq \ell_1<\cdots<\ell_t<a} \prod_{j=1}^{t} \frac{1}{(2\ell_j+1)^2}. \tag{1.8}$$

In terms of $w_t(a)$ and the numbers $c(\alpha,\beta,\gamma)$, we have the following explicit formulae for $\mathcal{U}_a(q)$.

**Theorem 1.3.** *If $a$ is a non-negative integer, then we have that*

$$\mathcal{U}_a(q) = \sum_{t=0}^{a} w_t(a) \sum_{\substack{\alpha,\beta,\gamma\geq 0\\ \alpha+2\beta+3\gamma=t}} c(\alpha,\beta,\gamma) E_2(q)^{\alpha} E_4(q)^{\beta} E_6(q)^{\gamma}.$$

*Example.* For $a = 3$, Theorem 1.3 gives

$$\mathcal{U}_3(q) = \frac{5}{7168} - \frac{37E_2(q)}{46080} + \frac{5E_2(q)^2}{27648} - \frac{E_4(q)}{13824} - \frac{E_2(q)^3}{82944} + \frac{E_2(q)E_4(q)}{69120} - \frac{E_6(q)}{181440}.$$

We turn to the $\mathcal{U}_a^*(q)$. Instead of using $E_2(q), E_4(q)$, and $E_6(q)$, we use all of the Eisenstein series

$$E_{2k}(q) := 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n, \tag{1.9}$$

where $B_k$ is the usual $k$th Bernoulli number. Namely, we let $\mathbb{E}_0(q) := 1$, and for positive $t$ we define

$$\mathbb{E}_{2t}^{\star}(q) := \sum_{(1^{m_1},\ldots,t^{m_t})\vdash t} \prod_{j=1}^{t} \frac{1}{m_j!} \left( -\frac{B_{2j}\, E_{2j}(q)}{(2j)\cdot(2j!)} \right)^{m_j}. \tag{1.10}$$

We require constants for the summands sorted by weight. We let $w_0^{\star}(0) := 1$, and for $a > 0$, we let

$$w_0^{\star}(a) := \sum_{i=1}^{a} \frac{(-1)^{i-1}\binom{2i}{i}}{16^i(2i+1)}\, w_0^{\star}(a-i). \tag{1.11}$$

For $1 \leq t \leq a$, we define

$$w_t^{\star}(a) := (-1)^{a+t-1} 4^t (2t+1)!\, w_{t-1}(a-1). \tag{1.12}$$

With this notation, we obtain the following explicit expressions for $\mathcal{U}_a^*(q)$.

**Theorem 1.4.** *If $a$ is a non-negative integer, then we have that*

$$\mathcal{U}_a^\star(q) = \sum_{t=0}^{a} w_t^\star(a) \cdot \mathbb{E}_{2t}^\star(q).$$

*Example.* For $a = 5$, Theorem 1.4 gives

$$\mathcal{U}_5^*(q) = \frac{1295803}{12262440960} + \frac{35}{294912}\mathbb{E}_2^\star(q) - \frac{3229}{967680}\mathbb{E}_4^\star(q) + \frac{47}{1152}\mathbb{E}_6^\star(q) - \frac{7}{24}\mathbb{E}_8^\star(q) + \mathbb{E}_{10}^\star(q).$$

The coefficients of $\mathcal{U}_a(q)$ and $U_a^\star(q)$ satisfy surprising congruences. Amdeberhan-Andrews-Tauraso [1] discovered some congruences that are reminiscent of Ramanujan's partition congruences, such as

$$MO(2; 5n+2) \equiv 0 \pmod 5 \qquad \text{and} \qquad MO(3; 7n+3) \equiv MO(3; 7n+5) \equiv 0 \pmod 7.$$

Moreover, they conjectured (see Conjecture 9.1 of [1]) that

$$MO(10; 11n+7) \equiv 0 \pmod{11}. \tag{1.13}$$

**Theorem 1.5.** *For every non-negative integer $n$, we have that*

$$MO(10; 11n+7) \equiv 0 \pmod{11}.$$

We offer two proofs of this result. The first proof uses the explicit description of $\mathcal{U}_{10}(q)$ provided by Theorem 1.3, which allows us to employ the "theory of modular forms mod $p$". This proof illustrates an algorithm that reduces the proof of all conjectured congruences of the form

$$MO(a; pn+r) \equiv 0 \pmod p \qquad \text{and} \qquad M(a; pn+r) \equiv 0 \pmod p$$

to finitely many steps. Theorem 1.5 requires computing at most 20 terms of five auxiliary $q$-series.

The second proof is a special case of one of three new infinite families of congruences.

**Theorem 1.6.** *The following are true:*
*(i) For every pair of non-negative integers $n$ and $m$, we have that*

$$MO(3m+2; 3n+1) \equiv MO(3m+2; 3n+2) \equiv 0 \pmod 3.$$

*(ii) For every pair of non-negative integers $n$ and $m$, we have*

$$MO(11m+10; 11n+7) \equiv 0 \pmod{11}.$$

*(iii) For every pair of non-negative integers $n$ and $m$, we have*

$$MO(17m+16; 17n+15) \equiv 0 \pmod{17}.$$

Computer searches for congruences suggest that such congruences are rare, thereby underscoring the significance of Theorem 1.5. However, it turns out that congruences are both rare and ubiquitous.

**Theorem 1.7.** *For positive integers $a$ and $m$, the following are true:*
*(i) There are infinitely many non-nested arithmetic progressions $tn + r$ (resp. $t^*n + r^*$) for which*

$$M(a; tn+r) \equiv 0 \pmod m,$$
$$MO(a; t^*n+r^*) \equiv 0 \pmod m.$$

*(ii) There are infinitely many non-nested arithmetic progressions $tn + r$ for which*

$$M(a; tn+r) \equiv MO(a; tn+r) \equiv 0 \pmod m.$$

*(iii) There exists a positive real number $\alpha(a,m)>0$ for which*

$$\#\{n\le X \;:\; M(a;n)\not\equiv 0 \pmod{m}\} = O\left(X/\log^{\alpha(a,m)} X\right)$$
$$\#\{n\le X \;:\; MO(a;n)\not\equiv 0 \pmod{m}\} = O\left(X/\log^{\alpha(a,m)X}\right).$$

*In other words, the values $M(a;n)$ and $MO(a;n)$ are almost always multiples of any integer $m$.*

To conclude, we offer infinite families of congruences when $a\in\{2,3,4,5\}$. For convenience, we let

$$N_a := \begin{cases} 2^3 & \text{if } a=2,\\ 2^7 3\cdot 5 & \text{if } a=3,\\ 2^{10}3^3\cdot 5\cdot 7 & \text{if } a=4,\\ 2^{15}3^3 5^2\cdot 7 & \text{if } a=5.\end{cases}$$

**Corollary 1.8.** *If $a\in\{2,3,4,5\}$, then the following are true:*
*(i) If $\ell\in\{2,3,5,7\}$ and $p\equiv -1 \pmod{\ell^{ord_\ell(N_a)+1}}$ is prime, then for every $n$ coprime to $p$ we have*

$$MO(a;pn)\equiv 0 \pmod{\ell}.$$

*(ii) If $\ell\ge 11$ is prime and $p\equiv -1 \pmod{\ell}$, then for every integer $n$ coprime to $p$ we have*

$$MO(a;pn)\equiv 0 \pmod{\ell}.$$

*Example.* The following congruence is an example of Corollary 1.8 $(i)$ :

$$MO(2,19^2n+19)\equiv MO(2,19^2n+38)\equiv MO(2,19^2n+57)\equiv MO(2,19^2n+76)\equiv 0 \pmod{5},$$

As an example of Corollary 1.8 $(ii)$, for $1\le t\le 18$, we have

$$MO(a,37^2n+37t)\equiv 0 \pmod{19}.$$

*Remark.* Most of the congruences in Theorem 1.7 do not belong to infinite families such as those in Corollary 1.8. For instance, if $p\in\{67,101,271,373\}$, then for every non-negative integer $n$ we have

$$M(6;pn)\equiv MO(6;pn)\equiv 0 \pmod{17}.$$

The coefficients of the expansion of $\mathcal{U}_6(q)$ provided by Theorem 1.3 are units modulo 17, and so these congruences follow from the fact that all of the monomials $E_2(q)^\alpha E_4(q)^\beta E_6(q)^\gamma$, with $\alpha,\beta,\gamma\ge 0$ and $\alpha+2\beta+3\gamma\le 6$, are annihilated modulo 17 by the Hecke operators $T_p$ for $p\in\{67,101,271,373\}$.

This paper is organized as follows. In Section 2, we recall the Nekrasov-Okounkov hook formulae and relevant results of Andrews-Rose and Amdeberhan-Andrews-Tauraso, which we then employ to prove Theorems 1.1 and 1.2 on the limiting behavior of $U_a(q)$ and $U_a^\star(q)$. In Section 3 we recall pertinent facts about symmetric functions, as well as results on the quasimodularity of $U_a(q)$, which we then use to prove Theorems 1.3 and 1.4. Finally, in Section 4 we prove Theorems 1.5 and 1.6, and in Section 5 we prove Theorem 1.7 using modularity.

## Acknowledgements

The second author thanks the Thomas Jefferson Fund and the NSF (DMS-2002265 and DMS-2055118). The third author is grateful for the support of a Fulbright Nehru Postdoctoral Fellowship.

## 2. Proofs of Theorems 1.1 and 1.2

Here we prove Theorems 1.1 and 1.2 using earlier work of Nekrasov-Okounkov and Andrews-Rose.

### 2.1. **Proof of Theorem 1.1.**

We require a beautiful identity of Andrews-Rose for $\mathcal{U}_a(q)$.

**Lemma 2.1.** [2, Cor. 2] *If $a$ is a positive integer, then as formal power series we have that*

$$\mathcal{U}_a(q)\cdot\prod_{n\geq 1}(1-q^n)^3=\frac{(-1)^a}{(2a+1)!}\sum_{n\geq 0}(-1)^n(2n+1)\frac{(n+a)!}{(n-a)!}\,q^{\frac{n(n+1)}{2}}.$$

We also require the celebrated Nekrasov-Okounkov hook length identity (see (6.12) on page 569 of [8]; see also Th. 1.3 of [5]).

**Theorem 2.2.** *As a formal power series, we have*

$$\prod_{j\geq 1}\frac{1}{(1-q^j)^{z+1}}=\sum_{m\geq 0}q^m\sum_{\lambda\vdash m}\prod_{h\in\mathcal{H}(\lambda)}\left(\frac{z}{h^2}+1\right).$$

*Proof of Theorem 1.1.* Thanks to Lemma 2.1 for $\mathcal{U}_a(q)$, we find that

$$\begin{aligned}q^{-\binom{a+1}{2}}\cdot\mathcal{U}_a(q)\cdot\prod_{n\geq 1}(1-q^n)^3&=\sum_{j\geq 0}(-1)^j\cdot\frac{2j+2a+1}{2a+1}\cdot\binom{j+2a}{j}\,q^{aj+\binom{j+1}{2}}\\&=1-(2a+3)q^{a+1}+(a+1)(2a+5)q^{2a+3}+\cdots.\end{aligned}$$

Claim $(i)$ follows immediately.

The first formula in $(ii)$ follows by letting $z=2$ in Theorem 2.2, giving

$$\prod_{n\geq 1}\frac{1}{(1-q^n)^3}=\sum_{m\geq 0}q^m\sum_{\lambda\vdash m}\prod_{h\in\mathcal{H}(\lambda)}\left(\frac{2}{h^2}+1\right),$$

while the other claim arises from the interpretation of the $q$-product in terms of 3-colored partitions. □

### 2.2. **Proof of Theorem 1.2.**

Amdeberhan-Andrews-Tauraso express $\mathcal{U}_a^\star(q)$ as a single sum.

**Lemma 2.3.** [1, Prop. 4.1] *We have the identity*

$$\mathcal{U}_a^\star(q)=\sum_{k\geq 1}(-1)^{k-1}\,\frac{(1+q^k)\,q^{\binom{k}{2}+ak}}{(1-q^k)^{2a}}.$$

*Proof of Theorem 1.2.* The expansion $(1-q^k)^{-2a}=\sum_{m\geq 0}\binom{2a+m-1}{m}q^{km}$ and Lemma 2.3 imply that

$$\begin{aligned}q^{-a}\cdot\mathcal{U}_a^\star(q)&=\sum_{k\geq 1}(-1)^{k-1}\,\frac{q^{\binom{k}{2}+a(k-1)}}{(1-q^k)^{2a}}+\sum_{k\geq 1}(-1)^{k-1}\,\frac{q^{\binom{k+1}{2}+a(k-1)}}{(1-q^k)^{2a}}\\&=\sum_{k\geq 1}\sum_{m\geq 0}(-1)^{k-1}\binom{2a+m-1}{m}q^{km+\binom{k}{2}+a(k-1)}+\sum_{k\geq 1}\sum_{m\geq 0}\binom{2a+m-1}{m}q^{km+\binom{k+1}{2}+a(k-1)}.\end{aligned}$$

We compare coefficients of $q^n$ for $n \leq a$. Namely, in the double sums we require $km+\binom{k}{2}+a(k-1) \leq a$ and $km + \binom{k+1}{2} + a(k-1) \leq a$. The former results in $k = 1, m = n$ and the latter forces $k = 1, m = n-1$. Consequently, if we let $q^{-a} \cdot \mathcal{U}_a^\star(q) =: \sum_{n\geq 0} p_n(a)\, q^n$, then we find that

$$p_n(a) = \binom{2a+n-1}{n} + \binom{2a+n-2}{n-1}.$$

□

## 3. Proof of Theorems 1.3 and 1.4

Here we prove the explicit descriptions of $\mathcal{U}_a(q)$ and $\mathcal{U}_a^*(q)$ in terms of Eisenstein series.

3.1. **Nuts and Bolts.** We make use of the differential operator $\Theta := q\frac{d}{dq}$, which acts by

$$\Theta\left(\sum a(n)q^n\right) := \sum na(n)q^n. \tag{3.1}$$

Ramanujan famously obtained the following formulas [11, p. 181] for the action of $\Theta$:

$$\Theta(E_2(q)) = \frac{E_2^2(q) - E_4(q)}{12}, \qquad \Theta(E_4(q)) = \frac{E_2(q)E_4(q) - E_6(q)}{3}, \tag{3.2}$$
$$\Theta(E_6(q)) = \frac{E_2(q)E_6(q) - E_4^2(q)}{2}.$$

The $q$-series $\mathcal{U}_a(q)$ and $\mathcal{U}_a^\star(q)$ satisfy the following convenient convolution (see [1, p. 13]).

**Lemma 3.1.** *If $a$ is a positive integer, then we have that*

$$\sum_{i=0}^{a} (-1)^i \cdot \mathcal{U}_i(q) \cdot \mathcal{U}_{a-i}^\star(q) = 0.$$

Recall the Dedekind eta-function $\eta(q) = q^{\frac{1}{24}} \prod_{m=1}^\infty (1-q^m)$. The following result of Rose [12, Th. 1.12] describes the structural framework of $\mathcal{U}_a(q)$ in terms of iterated derivatives of $\eta(q)^3$.

**Theorem 3.2.** *Each $\mathcal{U}_a(q)$ is a finite sum of quasimodular forms with weight $\leq 2a$ on $\mathrm{SL}_2(\mathbb{Z})$. Moreover, the weight $2t$ summand is a (possibly zero) scalar multiple of*

$$2^t \cdot \frac{\Theta^t\left(\eta(q)^3\right)}{\eta(q)^3}.$$

Our next result expresses these $q$-series as a linear combination of monomials $E_2(q)^\alpha E_4(q)^\beta E_6(q)^\gamma$.

**Lemma 3.3.** *If $t$ is a positive integer, then we have that*

$$(-8)^t \cdot \frac{\Theta^t\left(\eta(q)^3\right)}{\eta(q)^3} = \sum_{\substack{\alpha,\beta,\gamma \geq 0 \\ \alpha+2\beta+3\gamma = t}} c(\alpha,\beta,\gamma) \cdot E_2(q)^\alpha E_4(q)^\beta E_6(q)^\gamma$$

*where the coefficients $c(\alpha.\beta,\gamma)$ are defined by* (1.7).

*Proof.* For convenience, we let $\psi(q) := \eta(q)^3$. We calculate $\frac{\Theta^t(\psi(q))}{\psi(q)}$ by inducting on $t$. First, it is easy to check $\Theta(\psi(q)) = \frac{1}{8}\psi(q)E_2(q)$. Theorem 3.2 implies the existence of numbers $\widetilde{c}(\alpha,\beta,\gamma)$ for which

$$\frac{\Theta^t(\psi(q))}{\psi(q)} = \sum_{\substack{\alpha,\beta,\gamma\geq 0\\ \alpha+2\beta+3\gamma=t}} \widetilde{c}(\alpha,\beta,\gamma)\cdot E_2^{\alpha}(q)E_4^{\beta}(q)E_6^{\gamma}(q).$$

This comprises of all weight $2t$ quasimodular summands in $\mathcal{U}_a(q)$. One more derivative $\Theta = q\frac{d}{dq}$ turns the last equation into (for brevity, we write $\widetilde{c}$ in place of $\widetilde{c}(\alpha,\beta,\gamma)$)

$$\Theta^{t+1}(\psi(q)) = \Theta(\psi(q))\cdot\left(\sum_{\alpha,\beta,\gamma}\widetilde{c}\cdot E_2^{\alpha}(q)E_4^{\beta}(q)E_6^{\gamma}(q)\right) + \psi(q)\cdot\sum_{\alpha,\beta,\gamma}\widetilde{c}\cdot\Theta(E_2^{\alpha}(q)E_4^{\beta}(q)E_6^{\gamma}(q)).$$

On the other hand, Ramanujan's identities (3.2) imply that

$$\Theta(E_2^{\alpha}E_4^{\beta}E_6^{\gamma}) = \left(\frac{\alpha}{12}+\frac{\beta}{3}+\frac{\gamma}{2}\right)E_2^{\alpha+1}E_4^{\beta}E_6^{\gamma} - \frac{\alpha}{12}E_2^{\alpha-1}E_4^{\beta+1}E_6^{\gamma} - \frac{\beta}{3}E_2^{\alpha}E_4^{\beta-1}E_6^{\gamma+1} - \frac{\gamma}{2}E_2^{\alpha}E_4^{\beta+2}E_6^{\gamma-1}.$$

We find that the homogeneous weight $2t+2$ form satisfies

$$\begin{aligned}\frac{\Theta^{t+1}(\psi(q))}{\psi(q)} = &\sum_{\substack{\alpha,\beta,\gamma\geq 0\\ \alpha+2\beta+3\gamma=t}}\left(\frac{\alpha}{12}+\frac{\beta}{3}+\frac{\gamma}{2}+\frac{1}{8}\right)\widetilde{c}\cdot E_2^{\alpha+1}E_4^{\beta}E_6^{\gamma} - \sum_{\alpha,\beta,\gamma}\frac{\alpha}{12}\,\widetilde{c}\cdot E_2^{\alpha-1}E_4^{\beta+1}E_6^{\gamma}\\ &-\sum_{\alpha,\beta,\gamma}\frac{\beta}{3}\,\widetilde{c}\cdot E_2^{\alpha}E_4^{\beta-1}E_6^{\gamma+1} - \sum_{\alpha,\beta,\gamma}\frac{\gamma}{2}\,\widetilde{c}\cdot E_2^{\alpha}E_4^{\beta+2}E_6^{\gamma-1}.\end{aligned}$$

By comparing the coefficients of $E_2^{\alpha}E_4^{\beta}E_6^{\gamma}$ on both sides of the equation above, we obtain the recursion (with $\widetilde{c}(\alpha,\beta,\gamma) = \delta_{(0,0,0)}(\alpha,\beta,\gamma)$, a Dirac delta boundary conditions)

$$\begin{aligned}\widetilde{c}(\alpha,\beta,\gamma) = &\left(\frac{\alpha}{12}+\frac{\beta}{3}+\frac{\gamma}{2}+\frac{1}{24}\right)\widetilde{c}(\alpha-1,\beta,\gamma) - \frac{\alpha+1}{12}\cdot\widetilde{c}(\alpha+1,\beta-1,\gamma)\\ &-\frac{\beta+1}{3}\cdot\widetilde{c}(\alpha,\beta+1,\gamma-1) - \frac{\gamma+1}{2}\cdot\widetilde{c}(\alpha,\beta-2,\gamma+1).\end{aligned}$$

To determine the exact weight $2t$ term (independent of $a$), we take into account the factor of $(-8)^{\alpha+2\beta+3\gamma}$ to determine $c(\alpha,\beta,\gamma) := (-8)^{\alpha+2\beta+3\gamma}\cdot\widetilde{c}(\alpha,\beta,\gamma)$. As a result, we obtain the desired

$$\begin{aligned}c(\alpha,\beta,\gamma) = &-\frac{1}{3}\left(2\alpha+8\beta+12\gamma+1\right)\cdot c(\alpha-1,\beta,\gamma) + \frac{2}{3}(\alpha+1)\cdot c(\alpha+1,\beta-1,\gamma)\\ &+\frac{8}{3}(\beta+1)\cdot c(\alpha,\beta+1,\gamma-1) + 4(\gamma+1)\cdot c(\alpha,\beta-2,\gamma+1).\end{aligned}$$

□

3.2. **Proof of Theorem 1.3.** We let $\mathcal{E}_t(q) := (-8)^t\cdot\frac{\Theta^t(\psi)}{\psi}$, and we define

$$\mathbb{E}_{2t}(q) := \sum_{(1^{m_1},\ldots,t^{m_t})\vdash t}\prod_{j=1}^{t}\frac{1}{m_j!}\left(\frac{B_{2j}\,E_{2j}(q)}{(2j)\cdot(2j)!}\right)^{m_j}.$$

By inspection, we see that $\mathbb{E}_{2t}(q)$ has weight $2t$. We claim that

$$\mathbb{E}_{2t}(q) = \frac{(-1)^t}{4^t(2t+1)!} \cdot \mathcal{E}_t(q). \tag{3.3}$$

Let $\mathbf{S}_r(q) := \sum_{m\geq 1} \frac{m^r q^m}{1-q^m} = \sum_{n\geq 1} \sigma_r(n) q^n$. By expanding $\sum_{j,k\geq 1} \frac{q^{kj}\cos(2kx)}{k}$ in two different ways, we find that it equals both of these

$$\prod_{j\geq 1}\left[1 + \frac{4(\sin^2 x) q^j}{(1-q^j)^2}\right] = \exp\left(-2\sum_{r\geq 1} \frac{\mathbf{S}_{2r-1}(q)}{(2r)!}(-4x^2)^r\right). \tag{3.4}$$

Using the identity [1, p. 13], we obtain

$$\prod_{k\geq 1}\left(1 + \frac{4q^k \sin^2 x}{(1-q^k)^2}\right) = \sum_{a\geq 0} 4^a \mathcal{U}_a(q)(\sin x)^{2a}, \tag{3.5}$$

and the Jacobi Triple Product then implies that

$$\begin{aligned}
\sin x \prod_{k\geq 1}\left(1 + \frac{4q^k \sin^2 x}{(1-q^k)^2}\right) &= \frac{e^{ix}-e^{-ix}}{2i} \prod_{j\geq 1} \frac{(1-q^j e^{2ix})(1-q^j e^{-2ix})}{(1-q^j)^2} \\
&= \frac{1}{2i\cdot\psi(q)} \sum_{j\in\mathbb{Z}} (-1)^j q^{\binom{j+1}{2}} e^{(2n+1)ix} \\
&= \frac{1}{\psi(q)} \sum_{t\geq 0} (-1)^t \frac{x^{2t+1}}{(2t+1)!} \sum_{n\geq 0} (-1)^n (2n+1)^{2t+1} q^{\binom{n+1}{2}} \\
&= \sum_{t\geq 0} \mathcal{E}_t(q) \frac{x^{2t+1}}{(2t+1)!}. \qquad (3.6)
\end{aligned}$$

Using (1.9) and the generating function for Pólya's *cycle index formula* [10, (1,5)], we obtain

(3.7)
$$\sin x \cdot \exp\left(-2\sum_{r\geq 1} \frac{\mathbf{S}_{2r-1}(q)}{(2r)!}(-4x^2)^{2r}\right) = \sin x \cdot \frac{x}{\sin x} \cdot \sum_{t\geq 0}\left(\sum_{\lambda\vdash t}\prod_{j=1}^{t} \frac{1}{m_j!}\left(\frac{B_{2j}\cdot E_{2s}(q)}{(2j)\cdot(2j!}\right)^{m_j}\right)(-4x^2)^t.$$

Combining (3.4), (3.6), (3.7) and then comparing the coefficients of $x^{2t+1}$, we confirm (3.3).

To the complete the proof, it suffices to determine the constants $b_t(a)$ for which

$$\mathcal{U}_a(q) = \sum_{t=0}^{a} b_t(a) \cdot \mathbb{E}_{2t}(q). \tag{3.8}$$

It is convenient to recall the Andrews-Rose recursion [2, Cor. 3]

$$\mathcal{U}_a(q) = \frac{1}{2a(2a+1)}\left[(6\mathcal{U}_1(q) + a(a-1))\mathcal{U}_{a-1}(q) - 2\Theta(\mathcal{U}_{a-1}(q))\right]. \tag{3.9}$$

The structure of equation (3.8) is preserved by (3.9) because of the identity

$$\Theta(\mathbb{E}_{2t-2}) = t(2t+1)\mathbb{E}_{2t} - 3\mathbb{E}_2\mathbb{E}_{2t-2}.$$

It is straightforward to see that

$$b_t(a) = \frac{1}{8a(2a+1)}\left[(2a-1)^2 \cdot b_t(a-1) - 8t(2t+1)\cdot b_{t-1}(a-1)\right],$$

with initial boundary conditions $b_0(0) = 1$ and $b_t(a) = 0$ when $t < 0$ or $t > a$. Finally, one checks that $(-4)^t(2t+1)!\, w_t(a)$ satisfies this recurrence, thereby completing the proof of the theorem.

### 3.3. **Proof of Theorem 1.4.**

By reciprocating (3.5), we have

$$\sum_{n\geq 0}(-4)^a\,\mathcal{U}_a^\star(q)(\sin x)^{2a} = \prod_{k\geq 1}\frac{1}{1+\frac{4q^k\sin^2 x}{(1-q^k)^2}}.$$

In analogy with the previous formula for $\mathbb{E}_{2t}(q)$ involving the $\mathcal{U}_a(q)$, we use (3.3) to obtain an identity for $\mathcal{U}_a^\star(q)$ with $\mathbb{E}_{2t}^*(q)$ (see (1.10)). Arguing as in the proof of Theorem 1.3 with Lemma 3.1, we get

$$\mathcal{U}_a^\star(q) = \sum_{t=0}^{a} w_t^\star(a)\cdot \mathbb{E}_{2t}^\star(q).$$

## 4. Proof of Theorems 1.5 and 1.6

Here we prove Theorem 1.5 using Serre's theory of modular forms modulo primes $p$ (see [9, Section 2.8], or [17]) and a well-known criterion of Sturm that determines congruences between modular forms. In the sequel, we tacitly assume that $q := e^{2\pi i z}$, the uniformizer for the point at infinity. We also prove Theorem 1.6 by combining work of Andrews-Rose with a classical result of Gordon, together with other allied observations.

### 4.1. **Modular forms modulo $p$.**

We recall some facts from the theory of modular forms mod $p$. The key tool in the proof of Theorem 1.5 is the following theorem of Sturm (see [16] or p. 40 of [9]).

**Theorem 4.1.** *Let $p$ be a prime. If $f(z) = \sum_{n\geq 0} a(n)q^n$ and $g(z) = \sum_{n\geq 0} b(n)q^n$ are modular forms of weight $k$ on $\mathrm{SL}_2(\mathbb{Z})$ with integer coefficients, then $f(z) \equiv g(z) \pmod{p}$ if and only if $a(n) \equiv b(n) \pmod{p}$ for all $n \leq k/12$.*

We shall make use of derivatives of modular forms. Although differentiation does not preserve modularity, it does preserve modular forms modulo $p$ (for example, see [13]).

**Lemma 4.2.** *If $f(z) = \sum_{n\geq 0} a(n)q^n \in M_k \cap \mathbb{Z}[[q]]$, then there is a modular form $g(z) = \sum_{n\geq 0} b(n)q^n \in M_{k+p+1} \cap \mathbb{Z}[[q]]$ for which*

$$g \equiv \Theta(f) := \sum_{n\geq 0} na(n)q^n \pmod{p}.$$

### 4.2. **Proof of Theorem 1.5.**

We let $\mathcal{U}_{10}(q) = F_0(q) + F_2(q) + F_4(q) + F_6(q) + F_8(q)$, where each $F_{2i}(q)$ is a sum of $E_2^\alpha E_4^\beta E_6^\gamma$ (suppressing the $q$), where $2\alpha + 4\beta + 6\gamma \equiv 2i \pmod{10}$. Theorem 1.3

then gives

$$\begin{aligned}
F_0(q) =&\frac{46189}{5772436045824} - \frac{2008213E_4E_6}{4271802792542208000} + \cdots + \frac{E_2^{10}}{230078188847156428800},\\
F_2(q) =&-\frac{25587296781661E_2}{2645567198945303592960} - \frac{604841E_6^2}{480577814160998400000} + \cdots + \frac{7862933E_2^6}{639106080130990080000},\\
F_4(q) =&-\frac{79923511502753E_4}{671337541085744332800000} + \frac{79923511502753E_2^2}{268535016434297733120000} + \cdots - \frac{16333E_2^7}{447374256091693056000},\\
F_6(q) =&-\frac{70726885883E_6}{33320061781829222400000} + \frac{70726885883E_2E_4}{12693356869268275200000} - \cdots + \frac{1819E_2^8}{25564243205239603200000},\\
F_8(q) =&-\frac{316100258731E_4^2}{2073248288647151616000000} + \frac{316100258731E_2E_6}{388734054121340928000000} - \cdots - \frac{19E_2^9}{23007818884715642880000}.
\end{aligned}$$

Each of these $q$-series is 11-integral, and so they may be reduced modulo 11 to obtain

$$\begin{aligned}
\widehat{F}_0(q) &:= F_0(q) \pmod{11} \equiv 2q^3 + 6q^4 + 7q^5 + 8q^6 + 5q^7 + 2q^8 + 2q^9 + \dots &\pmod{11},\\
\widehat{F}_2(q) &:= F_2(q) \pmod{11} \equiv 6q^3 + 7q^4 + 10q^5 + 7q^6 + 8q^7 + 7q^8 + 6q^9 + \dots &\pmod{11},\\
\widehat{F}_4(q) &:= F_4(q) \pmod{11} \equiv 7q^3 + 10q^4 + 8q^5 + 2q^6 + 4q^8 + 7q^9 + \dots &\pmod{11},\\
\widehat{F}_6(q) &:= F_6(q) \pmod{11} \equiv 10q^3 + 8q^4 + 9q^5 + 10q^6 + q^7 + 4q^8 + 10q^9 + \dots &\pmod{11},\\
\widehat{F}_8(q) &:= F_8(q) \pmod{11} \equiv 8q^3 + 2q^4 + 6q^5 + 6q^6 + 8q^7 + 5q^8 + 8q^9 + \dots &\pmod{11}.
\end{aligned}$$

Using the congruences $E_2(q) \equiv E_{12}(q) \pmod{11}$ and $E_{10}(q) \equiv 1 \pmod{11}$, we observe that $\widehat{F}_0(q), \widehat{F}_2(q), \widehat{F}_4(q), \widehat{F}_6(q)$, and $\widehat{F_8}(q)$ are modular forms modulo 11 of weight $120, 72, 84, 96$, and 108, respectively, on $\mathrm{SL}_2(\mathbb{Z})$.

We proceed to isolate the arithmetic progression of coefficients that is relevant for the theorem. We apply the differential operators to $\mathcal{U}_{10}(q)$ to eliminate terms with exponents $n \equiv 0, 1, 3, 4, 5, 9 \pmod{11}$. The non-zero classes are the quadratic residues modulo 11. Using Fermat's Little Theorem and Euler's Criterion, this is achieved by

$$G_1(q) :\equiv \sum_{n\equiv 2,6,7,8,10 \pmod{11}} MO(10;n)q^n \equiv \sum_{i=0}^{4} -5[\Theta^{10}(\widehat{F}_{2i}(q)) - \Theta^5(\widehat{F}_{2i}(q))] \pmod{11}.$$

Next, we proceed to remove the terms with exponents that are quadratic non-residues apart from those with $n \equiv 7 \pmod{11}$. For instance, to eliminate $n \equiv 2 \pmod{11}$ from $G_1(q)$ compute

$$G_2(q) := \Theta(G_1(q)) - 2G_1(q) \equiv \sum_{n\equiv 6,7,8,10 \pmod{11}} MO(10;n)q^n \pmod{11}.$$

We repeat this process to remove terms with exponents $n \equiv 6, 8, 10 \pmod{11}$, and we get

$$\begin{aligned}\sum_{n\equiv 7 \pmod{11}} & MO(10;n)q^n \pmod{11} \\ &\equiv -5(\Theta^4(\widehat{F}_2) - \Theta^9(\widehat{F}_2)) + 9(\Theta^3(\widehat{F}_4) - \Theta^8(\widehat{F}_4)) - 3(\Theta^2(\widehat{F}_6) - \Theta^7(\widehat{F}_6)) + \Theta(\widehat{F}_8) \\ &\quad - \Theta^6(\widehat{F}_8) - 4(\widehat{F}_0 - \Theta^5(\widehat{F}_0)) - 5(\Theta^4(\widehat{F}_4) - \Theta^9(\widehat{F}_4)) + 9(\Theta^3(\widehat{F}_6) - \Theta^8(\widehat{F}_6)) \\ &\quad - 3(\Theta^2(\widehat{F}_8) - \Theta^7(\widehat{F}_8)) + \Theta(\widehat{F}_0) - \Theta^6(\widehat{F}_0) - 4(\widehat{F}_2 - \Theta^5(\widehat{F}_2)) - 5(\Theta^4(\widehat{F}_6) - \Theta^9(\widehat{F}_6)) \\ &\quad + 9(\Theta^3(\widehat{F}_8) - \Theta^8(\widehat{F}_8)) - 3(\Theta^2(\widehat{F}_0) - \Theta^7(\widehat{F}_0)) + \Theta(\widehat{F}_2) - \Theta^6(\widehat{F}_2) - 4(\widehat{F}_4 - \Theta^5(\widehat{F}_4)) \\ &\quad - 5(\Theta^4(\widehat{F}_8) - \Theta^9(\widehat{F}_8)) + 9(\Theta^3(\widehat{F}_0) - \Theta^8(\widehat{F}_0)) - 3(\Theta^2(\widehat{F}_2) - \Theta^7(\widehat{F}_2)) + \Theta(\widehat{F}_4) \\ &\quad - \Theta^6(\widehat{F}_4)4(\widehat{F}_6 - \Theta^5(\widehat{F}_6)) - 5(\Theta^4(\widehat{F}_0) - \Theta^9(\widehat{F}_0)) + 9(\Theta^3(\widehat{F}_2) - \Theta^8(\widehat{F}_2)) \\ &\quad - 3(\Theta^2(\widehat{F}_4) - \Theta^7(\widehat{F}_4)) + \Theta(\widehat{F}_6) - \Theta^6(\widehat{F}_6) - 4(\widehat{F}_8 - \Theta^5(\widehat{F}_8)).\end{aligned}$$

Now we collect these terms so that

$$\sum_{n\equiv 7 \pmod{11}} MO(10;n)q^n \pmod{11} = Y_0(q) + Y_2(q) + Y_4(q) + Y_6(q) + Y_8(q),$$

where $Y_{2i}(q)$ consists of those $\Theta^t(\widehat{F}_j(q))$ with weight congruent to $i$ modulo 10. By Lemma 4.2, we find that $Y_0(q), Y_2(q), Y_4(q), Y_6(q)$, and $Y_8(q)$ are modular forms modulo 11 with weights $228, 180, 192, 204$, and 216, respectively on $\mathrm{SL}_2(\mathbb{Z})$. Finally, by Sturm's Theorem 4.1 (i.e. checking at most 20 terms), we find that each of these modular forms vanishes modulo 11, which implies the theorem.

4.3. **Proof of Theorem 1.6.** The generating function for the 3-colored partition function satisfies

$$P_3(q) = \sum_{n\geq 0} p_3(n)q^n = \prod_{n\geq 1} \frac{1}{(1-q^n)^3} \equiv \sum_{n\geq 0} p(n)q^{3n} \pmod{3}.$$

Therefore, we have that $3 \mid p_3(n)$ whenever $3 \nmid n$. Furthermore, Gordon [4] proved that

$$p_3(11n+7) \equiv 0 \pmod{11}.$$

We further claim that $p_3(17n+15) \equiv 0 \pmod{17}$. To prove this congruence, we employ Ramanujan's weight 12 cusp form $\Delta(z) := \eta(z)^{24}$ through the following observation:

$$q^2 \sum_{n\geq 0} p_3(n)q^n \cdot \prod_{n\geq 1}(1-q^{17n})^3 \equiv q^2 \prod_{n\geq 1}(1-q^n)^{48} \pmod{17} = \Delta(z)^2.$$

One easily checks that $\Delta^2 \mid T_{17} \equiv 0 \pmod{17}$, which means that every seventeenth coefficient of $\Delta(z)^2$ vanishes modulo 17. This congruence follows immediately from the fact that

$$\prod_{n\geq 1}(1-q^{17n})^3 \in (\mathbb{Z}/17\mathbb{Z})[q^{17}].$$

Now suppose that $\ell \in \{3, 11, 17\}$ and $1 \leq a \equiv \ell - 1 \pmod{\ell}$. If $\ell \nmid n(n+1)/2$, then we have

$$\frac{(2n+1)(n+a)!}{(2a+1)!(n-a)!} \equiv 0 \pmod{\ell}.$$

Therefore, the identity in Lemma 2.1 collapses modulo $\ell$ and gives

$$\mathcal{U}_a(q) = \sum_{n\geq 0} MO(a;n)q^n \equiv P_3(q) \cdot \sum_{n\geq 0} A(a,\ell;\ell n)q^{\ell n} \pmod{\ell}.$$

The power series on the right is in $(\mathbb{Z}/\ell\mathbb{Z})[q^\ell]$, and so the $MO(a;n)$ inherit the $p_3(n)$ congruences.

## 5. Proof of Theorem 1.7

Here we prove Theorem 1.7 and Corollary 1.8.

5.1. **Nuts and Bolts.** We first recall the definition of Hecke operators. Let $m$ be a positive integer and $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_k$. Then the action of Hecke operator $T_m$ on $f(z)$ is defined by

$$f(z) \,|\, T_m := \sum_{n=0}^{\infty} \left( \sum_{d|\gcd(n,m)} d^{k-1} a\left(\frac{nm}{d^2}\right) \right) q^n. \tag{5.1}$$

In particular, if $m = p$ is a prime, we have

$$f(z) \,|\, T_p := f(z) \,|\, U_p + p^{k-1} f(z) \,|\, V_p, \tag{5.2}$$

where $f(z) \,|\, U_p := \sum_{n=0}^{\infty} a(pn)q^n$ and $f(z) \,|\, V_p := \sum_{n=0}^{\infty} a(n)q^{pn}$.

Let's recall a result of Serre [14] (also see [9, Lemma 2.63 and Theorem 2.65]) on the action of Hecke operator on cusp forms. For a number field $K$, let $\mathcal{O}_K$ denote its ring of integers.

**Lemma 5.1.** *For $1 \le i \le t$, let $f_i(z) = \sum_{n=1}^{\infty} a_i(n)q^n \in M_k$ be a modular form with coefficients in the ring of integers of a number field $O_K$. Then the following are true.*
*(i) If $\mathfrak{m} \subset \mathcal{O}_K$ is an ideal of norm $M$, then a positive proportion of the primes $p \equiv -1 \pmod{M}$ satisfy*

$$f_1(z) \mid T_p \equiv f_2(z) \mid T_p \equiv \cdots f_t(z) \mid T_p \equiv 0 \pmod{\mathfrak{m}}.$$

*(ii) There is a constant $a > 0$ such that for every $1 \le i \le t$ we have*

$$\#\left\{n \le X : a_i(n) \not\equiv 0 \pmod{\mathfrak{m}}\right\} = O\left(X/(\log X)^a\right).$$

We next recall some facts about $p$-adic modular forms developed by Serre [15]. Let $p$ be a prime. Consider the field of $p$-adic numbers $\mathbb{Q}_p$, with its non-archimedean valuation $\nu_p$. We say $x \in \mathbb{Q}_p$ is $p$-integral if $\nu_p(x) \ge 0$. Let $f = \sum a(n)q^n \in \mathbb{Q}_p[[q]]$ be a formal power series, we define $\nu_p(f) := \inf_n \nu_p(a_n)$. If $\nu_p(f) \ge m$, we write as well $f \equiv 0 \pmod{p^m}$. Assume $\{f_i\}$ to be a sequence of elements in $\mathbb{Q}_p[[q]]$. We say that $f_i \to f$ if the coefficients of $f_i$ tend uniformly to those of $f$, i.e., $\nu_p(f - f_i) \to \infty$. A *$p$-adic modular form* $f$ is a formal series with coefficients in $\mathbb{Q}_p$ which is the limit of classical modular forms $f_i$ of weights $k_i$.

In order to prove Theorem 1.7, we need the following preliminary result.

**Lemma 5.2.** *The following are true:*
*(i) If $m$ is a positive integer, then we have that*

$$E_2(z) \equiv \frac{1}{(2^m-1)} \sum_{i=1}^{m} 2^{i-1} E_{2+3\cdot 2^{m+1}}(z) \,|\, V_{2^{i-1}} \pmod{2^m}.$$

*Moreover, $E_2(z) \pmod{2^m}$ is the reduction of a weight $2 + 3 \cdot 2^{m+1}$ modular form on $\mathrm{SL}_2(\mathbb{Z})$.*
*(ii) If $m$ is a positive integer, then we have that*

$$E_2(z) \equiv \frac{2}{(3^m-1)} \sum_{i=1}^{m} 3^{i-1} E_{2+4\cdot 3^m}(z) \,|\, V_{3^{i-1}} \pmod{3^m}.$$

*Moreover,* $E_2(z) \pmod{3^m}$ *is the reduction of a weight* $2+4\cdot 3^m$ *modular form on* $\mathrm{SL}_2(\mathbb{Z})$*.*
*(iii) If* $p \geq 5$ *is prime and* $m$ *is a positive integer, then we have that*

$$E_2(z) \equiv \frac{(p-1)}{(p^m-1)} \sum_{i=1}^{m} p^{i-1} E_{2+(p-1)p^{m-1}}(z) \,|\, V_{p^{m-1}} \pmod{p^m}.$$

*In particular,* $E_2(z) \pmod{p^m}$ *is the reduction of a weight* $2+p^{m-1}(p-1)$ *modular form on* $\mathrm{SL}_2(\mathbb{Z})$*.*

*Proof.* Let $g(z)=\sum_{n=0}^{\infty} b(n)q^n$ be a weight $k$ modular form on $\mathrm{SL}_2(\mathbb{Z})$ with $p$-integral coefficients. Then $g(z)\,|\,T_p = \sum_{n=0}^{\infty} b(pn)q^n + p^{k-1}b(n)q^{pn} \in M_k$. Since $E_{p^{m-1}(p-1)}(z) \equiv 1 \pmod{p^m}$, we have that $\{g(z)E_{p^{m-1}(p-1)}(z)\}$ converges to $g(z)$ $p$-adically. Hence, $g(z)$ is a $p$-adic modular form. Also, we have the convergence

$$g(z)E_{p^{m-1}(p-1)}(z) \,|\, T_p \longrightarrow g(z) \,|\, U_p = \sum_{n=0}^{\infty} b(pn)q^n.$$

Hence, $U_p$ is an operator on $M_k$ and so is $V_p$, defined as $g(z)\,|\,V_p = p^{1-k}(g(z)\,|\,T_p - g(z)\,|\,U_p)$. Now, our proof of the lemma follows from [15, Example on Page 210]. □

5.2. **Proof of Theorem 1.7.** By Theorem 1.3 and 1.4, we have

$$\mathcal{U}_a(q) = \sum_{t=0}^{a} w_t(a) \sum_{\substack{\alpha,\beta,\gamma\geq 0\\ \alpha+2\beta+3\gamma=t}} c(\alpha,\beta,\gamma)E_2(q)^{\alpha}E_4(q)^{\beta}E_6(q)^{\gamma} = F_0(q)+F_2(q)+\cdots+F_{2a}(q),$$

$$\mathcal{U}^{\star}_a(q) = \sum_{t=0}^{a} w^{\star}_t(a)\cdot \mathbb{E}^{\star}_{2t}(q) = F^{\star}_0(q)+F^{\star}_2(q)+\cdots+F^{\star}_{2a}(q), \tag{5.3}$$

where $F_{2i}(q)$ and $F^{\star}_{2i}(q)$, for $0\leq i\leq a$, are quasimodular forms of weight $2i$ on $\mathrm{SL}_2(\mathbb{Z})$. Using Lemma 5.2 and the Chinese Remainder Theorem, we find that $F_{2i}(q)$ and $F^{\star}_{2i}(q)$, for $0\leq i\leq a$, are modular forms modulo any integer $m$ on $\mathrm{SL}_2(\mathbb{Z})$. Employing Lemma 5.1 $(i)$ on (5.3), we complete the proof of first and second parts of Theorem 1.7 and finally applying Lemma 5.1 $(ii)$ to (5.3), claim $(iii)$ follows.

5.3. **Proof of Corollary 1.8.** By Theorem 1.3, we find that

$$\begin{aligned}
\mathcal{U}_2(q) &= \frac{1}{2^3}\sum_{n\geq 0}[(-2n+1)\sigma_1(n)+\sigma_3(n)]q^n,\\
\mathcal{U}_3(q) &= \frac{1}{2^7\cdot 3\cdot 5}\sum_{n\geq 0}[(40n^2-100n+37)\sigma_1(n)+(-30n+50)\sigma_3(n)+3\sigma_5(n)]q^n,\\
\mathcal{U}_4(q) &= \frac{1}{2^{10}3^3\cdot 5\cdot 7}\sum_{n\geq 0}\big[(-840n^3+5880n^2-9870n+3229)\sigma_1\\
&\qquad +(756n^2-4410n+4935)\sigma_3+(-126n+441)\sigma_5+5\sigma_7\big]q^n,\\
\mathcal{U}_5(q) &= \frac{1}{2^{15}3^3 5^2\cdot 7}\sum_{n\geq 0}[(3360n^4-50400n^3+223440n^2-314200n+96111)\sigma_1(n)\\
&\qquad +(-3360n^3+45360n^2-167580n+157100)\sigma_3(n)\\
&\qquad +(720n^2-7560n+16758)\sigma_5(n)+(-50n+300)\sigma_7(n)+\sigma_9(n)]q^n.
\end{aligned}$$

Let $s$ and $t$ be non-negative integers. If $k$ is a positive odd, then for primes $p \equiv -1 \pmod{s}$ we have

$$(pn)^t \sigma_k(pn) = (pn)^t \sigma_k(n)\sigma_k(p) = (pn)^t \sigma_k(n)(1+p^k) \equiv 0 \pmod{s} \tag{5.4}$$

for all $n$ coprime to $p$. Corollary 1.8 follows by applying (5.4) appropriately in each case.

## References


[1] T. Amdeberhan, G. E. Andrews, R. Tauraso, *Extensions of MacMahon's sums of divisors*, preprint available at https://arxiv.org/pdf/2309.03191.pdf (2023).

[2] G. E. Andrews, S. C. F. Rose, *MacMahon's sum-of-divisors functions, Chebyshev polynomials, and quasimodular forms*, J. Reine Angew. Math. **676** (2013), 97–103.

[3] B. Brindle, *A unified approch to qMZVs*, preprint available at https://arxiv.org/pdf/2111.00051.pdf (2021).

[4] B. Gordon, *Ramanujan congruences for* $p_{-k} \pmod{11^r}$, Glasgow J. Math. **24** (1983), 107-123.

[5] G.-H. Han, *The Nekrasov-Okounkov hook length formula: refinement, elementary proof, extension and applications*, Annales de l'Institut Fourier, Volume 60 (2010) no. 1, 1-29.

[6] M. Kaneko, D. Zagier, *A generalized Jacobi theta function and quasimodular forms*, The moduli space of curves (Texas Island, 1994), Progr. Math., vol. **129**, Birkhäuser Boston, Boston, MA, 1995, 165–172.

[7] P. A. MacMahon, *Divisors of Numbers and their Continuations in the Theory of Partitions*, Proc. London Math. Soc. (2) **19** (1920), no.1, 75-113 [also in Percy Alexander MacMahon Collected Papers, Vol.2, pp. 303–341 (ed. G.E. Andrews), MIT Press, Cambridge, 1986].

[8] N. A. Nekrasov, A. Okounkov, *Seiberg-Witten theory and random partitions, in The unity of mathematics*, Progress in Mathematics, Birkhäuser Boston, 2006, vol. 244, 525-596.

[9] K. Ono, *The web of modularity: arithmetic of the coefficients of modular forms and q-series*, CBMS Regional Conference Series in Mathematics, **102**, Amer. Math. Soc., Providence, RI, 2004.

[10] G. Pólya, *Kombinatorische Anzahlbestimmungen für Gruppen, Graphen und chemische Verbindungen*, Acta Mathematica, **68** (0), (1937), 145-254.

[11] S. Ramanujan, *On certain arithmetical functions*, Trans. Camb. Phil. Soc., **22** (1916), 159–184.

[12] S. C. F. Rose, *Quasimodularity of generalized sum-of-divisors functions*, Research in Number Theory **1** (2015), Paper No. 18, 11 pp.

[13] J.-P. Serre, *A course in arithmetic*, Springer-Verlag, New York, 1973.

[14] J.-P. Serre, *Divisibilité de certaines fonctions arithmétiques*, L'Enseignement Math. **22** (1976), 227–260.

[15] J.-P. Serre, *Formes modulaires et fonctions zêta p-adiques*, Springer Lect. Notes **350** (1973), 191–268.

[16] J. Sturm, *On the congruence of modular forms*, Springer Lect. Notes in Math. **1240** (1984), 275–280.

[17] H. P. F. Swinnerton-Dyer, *On $\ell$-adic representations and congruences for coefficients of modular forms*, Springer Lect. Notes. **350** (1973), 1–55.



Dept. of Mathematics, Tulane University, New Orleans, LA 70118
*Email address*: tamdeber@tulane.edu

Dept. of Mathematics, University of Virginia, Charlottesville, VA 22904
*Email address*: ko5wk@virginia.edu
*Email address*: ajit18@iitg.ac.in