GENERALIZING A PUTNAM 2014 QUESTION

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07 December, 2014

Yesterday, 06 December 2014, held the Putnam Math Competition throughout the U.S.A. One of the problems in the morning session is about a determinant evaluation.

Problem A2. *Find a closed form for the determinant*

$$\det\left(\frac{1}{\min(i,j)}\right)_{i,j=1}^{n}.$$

Of course, a careful elementary row and column expansions would yield the value $(-1)^{n-1}\frac{n}{n!^2}$. Herewith, we generalize and prove the result in a unified and simpler way.

Generalization. *Suppose $a,b\in\mathbb{N}$. We have*

$$\det\left(\frac{1}{x_{\min(i+a,j+b)}}\right)_{i,j=1}^{n}=\begin{cases}\frac{1}{x_{\min(a+1,b+1)}} & \text{if } n=1 \text{ and } a\neq b\\ 0 & \text{if } n>1 \text{ and } a\neq b\\ \frac{1}{x_{n+a}}\prod_{i=1}^{n-1}\frac{x_{i+a}-x_{i+a+1}}{x_{i+a}^2} & \text{if } a=b.\end{cases}$$

Proof. Denoting the left-hand side by $M_n(a,b)$. The proof can readily be executed (inductively) by the Dodgson's Condensation method in the form

$$M_n(a,b)=\frac{M_{n-1}(a,b)M_{n-1}(a+1,b+1)-M_{n-1}(a+1,b)M_{n-1}(a,b+1)}{M_{n-2}(a+1,b+1)}.$$

The reader is advised to consider the 3 different cases, separately. □

The following identity appeared in a paper by T. Mansour and Y. Sun ("Dyck paths and partial Bell polynomials") where it is stated for $n=pk+\ell$ where $0\le\ell\le k-1$. We generalize and provide a proof with the WZ methodology.

Proposition. *For any $n,k,\ell\in\mathbb{N}$, we have*

$$\sum_{m=0}^{\min(n,p)}\frac{n+m\ell-mk}{m+1}\binom{n}{m}\binom{p}{k}=\frac{n(p(\ell+1)+n-pk+1)}{(n+1)(p+1)}\binom{n+p}{n}.$$

Proof. Divide the summand on the left side by the right-hand side to denote by $F(n,m)$. Zeilberger's algorithm provides the recurrence $F(n+1,k)-F(n,k)=G(n,k+1)-G(n,k)$ where $G(n,k)=R(n,k)F(n,k)$ with the rational function $R(n,k)=\frac{m(m+1)P(n,k)}{Q(n,k)}$ as a *certificate* given by

$$\begin{aligned}P(n,k)&=k-\ell+n+pn-nkp+n\ell p+n^2+2\ell k-2km-2p\ell km+2m\ell+nm\ell-nkm-\ell p+kp\\&\quad-p\ell^2-pk^2+p\ell m-pkm+p\ell^2m+pk^2m+2p\ell k+k^2m+\ell^2m-k^2-\ell^2-2\ell km,\end{aligned}$$

and $Q(n,k)=(-p+m-1)(-n-m\ell+km)(-n-p-2+kp+k-\ell p-\ell)(n+p+1)$. □