PROOF OF FORMULA 3.198

$$\int_0^1 x^{\mu-1}(1-x)^{\nu-1}\left[ax+b(1-x)+c\right]^{-(\mu+\nu)}\, dx = \frac{B(\mu,\nu)}{(a+c)^{\mu}\,(b+c)^{\nu}}$$

Start with

$$\int_0^1 \frac{x^{\mu-1}(1-x)^{\nu-1}}{\left[ax+b(1-x)+c\right]^{\mu+\nu}}\, dx = (b+c)^{-\mu-\nu}\int_0^1 x^{\mu-1}(1-x)^{\nu-1}\left[1-\frac{b-a}{b+c}x\right]^{-(\mu+\nu)}\, dx.$$

The integral representation for the hypergeometric function

$${}_2F_1\left[\alpha,\beta;\gamma;z\right] = \frac{1}{B(\beta,\alpha-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}\, dt$$

gives

$$\int_0^1 \frac{x^{\mu-1}(1-x)^{\nu-1}\, dx}{\left[ax+b(1-x)+c\right]^{\mu+\nu}} = (b+c)^{-\mu-\nu}B(\mu,\nu){}_2F_1\left[\mu+\nu,\mu;\mu+\nu;\frac{b-a}{b+c}\right].$$

The identity

$${}_2F_1[\alpha,\beta;\alpha;z] = (1-z)^{-\beta},$$

produces the final result.