PROOF OF FORMULA 3.411.26

$$\int_0^\infty xe^{-x} \frac{1-e^{-x}}{1+e^{-3x}}\, dx = \frac{2\pi^2}{27}$$

Define

$$J_n := \int_0^\infty \frac{xe^{-nx}\, dx}{1+e^{-3x}},$$

so that the requested evaluation is $J_1 - J_2$.

The change of variables $t = 3x$ and expanding the integrand in series gives

$$J_1 = \frac{1}{9} \sum_{k=0}^\infty (-1)^k \int_0^\infty te^{-(k+1/3)t}\, dt.$$

The change of variables $u = (k+1/3)t$ gives

$$J_1 = \sum_{k=0}^\infty \frac{(-1)^k}{(3k+1)^2}.$$

The same type of argument produces

$$J_2 = \sum_{k=0}^\infty \frac{(-1)^k}{(3k+2)^2}.$$

It follows that

$$\begin{aligned}
\int_0^\infty xe^{-x} \frac{1-e^{-x}}{1+e^{-3x}}\, dx = J_1 - J_2 &= \sum_{k=0}^\infty \frac{(-1)^k}{(3k+1)^2} - \sum_{k=0}^\infty \frac{(-1)^k}{(3k+2)^2} \\
&= -\sum_{k=0}^\infty \frac{(-1)^{3k+1}}{(3k+1)^2} - \sum_{k=0}^\infty \frac{(-1)^{3k+2}}{(3k+2)^2} \\
&= -\sum_{k=1}^\infty \frac{(-1)^k}{k^2} + \sum_{k=1}^\infty \frac{(-1)^{3k}}{(3k)^2} \\
&= -\frac{8}{9} \sum_{k=1}^\infty \frac{(-1)^k}{k^2}.
\end{aligned}$$

The value

$$\sum_{k=1}^\infty \frac{(-1)^k}{k^2} = -\frac{\pi^2}{12}$$

gives the result.