PROOF OF FORMULA 3.411.28

$$\int_0^\infty \frac{e^{-\nu x} - e^{-\mu x}}{e^{-x}+1} \frac{dx}{x} = \ln \left(\frac{\Gamma(\frac{\nu}{2})\,\Gamma(\frac{\mu+1}{2})}{\Gamma(\frac{\mu}{2})\,\Gamma(\frac{\nu+1}{2})} \right)$$

Observe that

$$e^{-\nu x} - e^{-\mu x} = -x \int_\mu^\nu e^{-xt} dt.$$

Therefore

$$\int_0^\infty \frac{e^{-\nu x} - e^{-\mu x}}{e^{-x}+1} \frac{dx}{x} = -\int_\mu^\nu \int_0^\infty \frac{e^{-xt}\, dx}{e^{-x}+1}\, dt.$$

The change of variable $y = e^{-t}$ yields

$$\int_0^\infty \frac{e^{-\nu x} - e^{-\mu x}}{e^{-x}+1} \frac{dx}{x} - \int_\mu^\nu \int_0^1 \frac{y^{t-1}\, dy}{1+y}\, dt.$$

The inner integral is obtained in terms of the β-*function* defined by

$$\beta(t) = \frac{1}{2}\psi\left(\frac{t+1}{2}\right) - \frac{1}{2}\psi\left(\frac{t}{2}\right),$$

where $\psi(x)$ is the *digamma function* defined by $\psi(x) = \Gamma'(x)/\Gamma(x)$. The definition of the β-function appears in **8.370** and its integral representation

$$\beta(x) = \int_0^1 \frac{t^{x-1}\, dt}{1+t}$$

is **8.371.1**.

Then the original integral is expressed as

$$-\frac{1}{2}\int_\mu^\nu \psi\left(\frac{t+1}{2}\right) dt + \frac{1}{2}\int_\mu^\nu \psi\left(\frac{t}{2}\right) dt = -\int_{(\mu+1)/2}^{(\nu+1)/2} \psi(x)\, dx + \int_{\mu/2}^{\nu/2} \psi(x)\, dx.$$

The result now follows from the relation

$$\psi(x) = \frac{d}{dx} \ln \Gamma(x).$$