PROOF OF FORMULA 3.411.29

$$\int_{-\infty}^{\infty} \frac{e^{px} - e^{qx}}{1+e^{rx}} \, \frac{dx}{x} = \ln \left(\tan \frac{\pi p}{2r} \, \cot \frac{\pi q}{2r} \right)$$

The change of variables $t = e^{rx}$ gives (assuming $r > 0$)

$$\int_{-\infty}^{\infty} \frac{e^{px} - e^{qx}}{1+e^{rx}} \, \frac{dx}{x} = \int_0^{\infty} \frac{t^{p/r-1} - t^{q/r-1}}{(1+t) \, \ln t} \, dt.$$

Split the integral into the two intervals $[0, 1]$ and $[1, \infty)$ and make the change of variables $s = 1/t$ in the second interval to produce

$$\int_0^{\infty} \frac{t^{p/r-1} - t^{q/r-1}}{(1+t) \, \ln t} \, dt = \int_0^{1} \frac{t^{p/r-1} - t^{q/r-1}}{(1+t) \, \ln t} \, dt + \int_0^{1} \frac{t^{-q/r} - t^{-p/r}}{(1+t) \, \ln t} \, dt.$$

These two integrals can be evaluated using entry **4.267.9**:

$$\int_0^1 \frac{x^{a-1} - x^{b-1}}{(1+x) \, \ln x} \, dx = \ln \left[\frac{\Gamma(b/2) \, \Gamma((a+1)/2)}{\Gamma(a/2) \, \Gamma((b+1)/2)} \right]$$

to get the result.