PROOF OF FORMULA 3.451.1

$$\int_0^\infty xe^{-x}\sqrt{1-e^{-x}}\,dx = \frac{4}{9}(4-3\ln 2)$$

Observe that

$$\int_0^\infty xe^{-x}\sqrt{1-e^{-x}}\,dx = -h'(1),$$

where

$$h(a) = \int_0^\infty e^{-ax}\sqrt{1-e^{-x}}\,dx.$$

The change of variables $t = e^{-x}$ gives

$$h(a) = \int_0^1 t^{a-1}(1-t)^{1/2}\,dt = B(a, \tfrac{3}{2}).$$

Differentiation yields

$$h'(a) = h(a)\left[\psi(a) - \psi(a+\tfrac{3}{2})\right],$$

where $\psi = \Gamma'/\Gamma$ is the polygamma fnction. Therefore

$$\int_0^\infty xe^{-x}\sqrt{1-e^{-x}}\,dx = -\frac{\Gamma(1)\Gamma(\frac{3}{2})}{\Gamma(\frac{5}{2})}\left[\psi(1) - \psi\left(\tfrac{5}{2}\right)\right].$$

The value $\psi(1) = -\gamma$ and $\psi(\frac{5}{2}) = -\gamma - 2\ln 2 + \frac{8}{3}$ give the result. To obtain the expression for $\psi(\frac{5}{2})$, differentiate the duplication formula

$$\Gamma(2z) = \frac{2^{2x-1}}{\sqrt{\pi}}\Gamma(z)\Gamma(z+\tfrac{1}{2}),$$

to produce

$$2\psi(2z) = 2\ln 2 + \psi(z) + \psi(z+\tfrac{1}{2}).$$

The simplification also makes use of the relation

$$\psi(n) = -\gamma + \sum_{k=1}^{n-1}\frac{1}{k}.$$