PROOF OF FORMULA 3.452.3

$$\int_0^\infty \frac{xe^{-x}\,dx}{\sqrt{e^x-1}} = \frac{\pi}{2}\left(2\ln 2 - 1\right)$$

Write the problem as

$$\int_0^\infty \frac{xe^{-x}\,dx}{\sqrt{e^x-1}} = \int_0^\infty \frac{x\,e^{-3x/2}\,dx}{\sqrt{1-e^{-x}}}.$$

Now define

$$f(a) := \int_0^\infty \frac{e^{-ax}\,dx}{\sqrt{1-e^{-x}}},$$

so that

$$f'(a) = -\int_0^\infty \frac{xe^{-ax}\,dx}{\sqrt{1-e^{-x}}}.$$

Then

$$\int_0^\infty \frac{xe^{-x}\,dx}{\sqrt{e^x-1}} = -f'(\tfrac{3}{2}).$$

Now let $u = e^{-x}$ to obtain

$$f(a) = \int_0^1 u^{a-1}(1-u)^{-1/2}\,du = B(a,\tfrac{1}{2}) = \frac{\Gamma(a)\,\Gamma(\frac{1}{2})}{\Gamma(a+\frac{1}{2})}.$$

The integral representation for the beta function

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt$$

was used. Logarithmic differentiation gives

$$f'(a) = f(a)\left[\psi(a) - \psi(a+\tfrac{1}{2})\right].$$

The values $f(\frac{3}{2}) = \frac{\pi}{2}$ and

$$\psi(n) = -\gamma + \sum_{k=1}^{n-1}\frac{1}{k} \text{ and } \psi(n+\tfrac{1}{2}) = -\gamma + 2\left(\sum_{k=1}^{n}\frac{1}{2k-1} - \ln 2\right),$$

give

$$\psi(\tfrac{3}{2}) = 2 - \gamma - 2\ln 2 \text{ and } \psi(2) = 1-\gamma.$$

Therefore

$$f'(\tfrac{3}{2}) = \frac{\pi}{2}\left((2-\gamma-2\log 2) - (1-\gamma)\right) = \frac{\pi}{2}(1-2\log 2).$$

This completes the evaluation.