PROOF OF FORMULA 3.452.5

$$\int_0^\infty \frac{xe^{-2x} \, dx}{\sqrt{e^x - 1}} = \frac{\pi}{16} \left(12 \ln 2 - 7 \right)$$

Write the problem as

$$\int_0^\infty \frac{xe^{-2x} \, dx}{\sqrt{e^x - 1}} = \int_0^\infty \frac{x \, e^{-5x/2} \, dx}{\sqrt{1 - e^{-x}}}.$$

Now define

$$f(a) := \int_0^\infty \frac{e^{-ax} dx}{\sqrt{1 - e^{-x}}},$$

so that

$$f'(a) = -\int_0^\infty \frac{xe^{-ax} dx}{\sqrt{1 - e^{-x}}}.$$

Then

$$\int_0^\infty \frac{xe^{-x} \, dx}{\sqrt{e^x - 1}} = -f'(\frac{5}{2}).$$

Now let $u = e^{-x}$ to obtain

$$f(a) = \int_0^1 u^{a-1} (1-u)^{-1/2} du = B(a, \frac{1}{2}) = \frac{\Gamma(a) \Gamma(\frac{1}{2})}{\Gamma(a + \frac{1}{2})}.$$

The integral representation for the beta function

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt$$

was used. Logarithmic differentiation gives

$$f'(a) = f(a) \left[\psi(a) - \psi(a + \frac{1}{2}) \right].$$

The values $f(\frac{5}{2}) = \frac{3\pi}{8}$ and

$$\psi(n) = -\gamma + \sum_{k=1}^{n-1} \frac{1}{k} \text{ and } \psi(n + \frac{1}{2}) = -\gamma + 2\left(\sum_{k=1}^{n} \frac{1}{2k-1} - \ln 2\right),$$

give

$$\psi(\frac{5}{2}) = \frac{8}{3} - \gamma - 2 \ln 2$$
 and $\psi(3) = \frac{3}{2} - \gamma$.

Therefore

$$f'(\frac{3}{2}) = \frac{3\pi}{8} \left(\left(\frac{8}{3} - \gamma - 2\log 2 \right) - \left(\frac{3}{2} - \gamma \right) \right) = -\frac{3\pi}{4} \left(\frac{7}{12} - \ln 2 \right).$$

This completes the evaluation.