PROOF OF FORMULA 3.522.1

$$\int_0^\infty \frac{x\,dx}{(b^2+x^2)\,\sinh ax} = \frac{\pi}{2ab} + \pi \sum_{k=1}^{\infty} \frac{(-1)^k}{ab+\pi k}$$

The change of variables $x = bt$ shows that is suffices to assume $b = 1$. Extend the integrand f to the whole real line using its symmetry. The integrand has poles in the upper half plane at $z = i$ and $z = \pi k i/a$. Assume first that $a \neq \pi k$ so these poles are simple.

Integrate over a semi-circle centered at the origin and radius R. The residues are

$$\begin{aligned} \text{Res}(f;i) &= \frac{1}{4\sinh(ia)} = \frac{1}{4i\sin a}, \\ \text{Res}(f;\pi ik/a) &= \frac{(-1)^k \pi ik}{2(a^2-\pi^2k^2)}. \end{aligned}$$

The residue theorem and a partial fraction decomposition give the value

$$\int_0^\infty \frac{x\,dx}{(b^2+x^2)\,\sinh ax} = \frac{\pi}{2}\left(\frac{1}{\sin a} - \sum_{k=1}^{\infty} \frac{(-1)^k}{a-\pi k} + \sum_{k=1}^{\infty} \frac{(-1)^k}{a+\pi k} \right)$$

and the result follows from the expansion

$$\frac{1}{\sin a} = \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{a+\pi k}.$$

The case of $a = \pi k$, thay yields a double pole, is treated as a limiting case with the same result.