PROOF OF FORMULA 3.541.7

$$\int_0^\infty e^{-\mu x} \tanh x \, dx = \beta\left(\frac{\mu}{2}\right) - \frac{1}{\mu}$$

Write the integral as

$$\int_0^\infty e^{-\mu x} \tanh x \, dx = \int_0^\infty e^{-\mu x} \frac{1 - e^{-2x}}{1 + e^{-2x}} \, dx$$

and let $t = e^{-2x}$ to ontain

$$\int_0^\infty e^{-\mu x} \tanh x \, dx = \frac{1}{2} \int_0^1 \frac{t^{\mu/2-1} - t^{\mu/2}}{1+t} \, dt.$$

The β function is defined by the integral representation

$$\beta(x) = \int_0^1 \frac{t^{x-1} \, dt}{1+t}$$

that appears as entry **8.371.1**. Therefore

$$\int_0^\infty e^{-\mu x} \tanh x \, dx = \frac{1}{2}\beta\left(\frac{\mu}{2}\right) - \frac{1}{2}\beta\left(\frac{\mu}{2} + 1\right).$$

From the identity

$$\beta(x) = \frac{1}{2}\left[\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right)\right],$$

it follows that

$$\int_0^\infty e^{-\mu x} \tanh x \, dx = \frac{1}{4}\left[\psi\left(\frac{\mu}{4} + \frac{1}{2}\right) - \psi\left(\frac{\mu}{4}\right) - \psi\left(\frac{\mu}{4} + 1\right)\right].$$

Now use $\psi(x+1) = \psi(x) + \frac{1}{x}$ to obtain the result.