PROOF OF FORMULA 3.611.1

$$\int_0^{2\pi} (1-\cos x)^n \sin nx \, dx = 0$$

Define

$$I_m = \int_0^{\pi} \sin^m t \sin mt \, dt$$

and integrate by parts to obtain

$$I_m = \int_0^{\pi} \sin^{m-1} t \, \cos mt \, \cos t \, dt.$$

The identity $\cos(m-1)t = \cos mt \cos t + \sin mt \sin t$ yields

$$2I_m = J_{m-1} := \int_0^{\pi} \sin^{m-1} t \cos(m-1)t \, dt.$$

A similar argument yields $2J_m = -I_{m-1}$. Therefore

$$I_m = -\frac{1}{4} I_{m-2} \text{ and } J_m = -\frac{1}{4} J_{m-2}.$$

The recurrence shows that $I_{2n} = (-1)^n I_0/2^{2n}$. The value $I_0 = 0$ gives $I_{2n} = 0$.

To evaluate the integral presented here, write $1 - \cos x = 2\sin^2(x/2)$ to produce

$$\int_0^{2\pi} (1-\cos x)^n \sin nx \, dx = 2^{n+1} \int_0^{\pi} \sin^{2n} t \, \sin 2nt \, dt = 2^{n+1} I_{2n} = 0,$$

as claimed.