PROOF OF FORMULA 3.611.2

$$\int_0^{2\pi} (1-\cos x)^n \cos nx \, dx = (-1)^n \frac{\pi}{2^{n-1}}$$

Define

$$J_m = \int_0^{\pi} \sin^m t \cos mt \, dt$$

and integrate by parts to obtain

$$J_m = -\int_0^{\pi} \sin^{m-1} t \, \sin mt \, \cos t \, dt.$$

The identity $\sin(m-1)t = \sin mt \cos t - \cos mt \sin t$ yields

$$2J_m = -I_{m-1} := \int_0^{\pi} \sin^{m-1} t \sin(m-1)t \, dt.$$

A similar argument yields $2I_m = J_{m-1}$. Therefore

$$J_m = -\frac{1}{4} J_{m-2} \text{ and } I_m = -\frac{1}{4} I_{m-2}.$$

The recurrence shows that $J_{2n} = (-1)^n J_0/2^{2n}$ and $J_0 = \pi$ produces

$$J_{2n} = (-1)^n \pi/2^{2n}.$$

To evaluate the integral presented here, write $1 - \cos x = 2\sin^2(x/2)$ to get

$$\int_0^{2\pi} (1-\cos x)^n \cos nx \, dx = 2^{n+1} \int_0^{\pi} \sin^{2n} t \, \cos 2nt \, dt = 2^{n+1} J_{2n} = (-1)^n \pi/2^{n-1},$$

as claimed.