PROOF OF FORMULA 4.221.2

$$\int_0^1 \ln x \, \ln(1+x)\, dx = 2 - \frac{\pi^2}{12} - 2\ln 2$$

Integrating the expansion

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k,$$

gives

$$\int_0^1 \ln x \, \ln(1+x)\, dx = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \int_0^1 x^k \ln x \, dx.$$

The change of variables $u = -\ln x$ gives

$$\int_0^1 x^k \ln x \, dx = -\int_0^{\infty} u e^{-(k+1)u}\, du$$

and $v = (k+1)u$ gives

$$\int_0^{\infty} u e^{-(k+1)u}\, du = -\frac{1}{(k+1)^2} \int_0^{\infty} v e^{-v}\, dv.$$

This last integral is $\Gamma(2) = 1$. Therefore

$$\int_0^1 \ln x \, \ln(1+x)\, dx = \sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)^2}.$$

The partial fraction decomposition

$$\frac{1}{k(k+1)^2} = \frac{1}{k} - \frac{1}{k+1} - \frac{1}{(k+1)^2},$$

gives

$$\int_0^1 \ln x \, \ln(1+x)\, dx = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} - \sum_{k=1}^{\infty} \frac{(-1)^k}{k+1} - \sum_{k=1}^{\infty} \frac{(-1)^k}{(k+1)^2}.$$

The values

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} = -\ln 2 \text{ and } \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} = -\frac{\pi^2}{12},$$

complete the evaluation.