PROOF OF FORMULA 4.221.3

$$\int_0^1 \ln\left(\frac{1-ax}{1-a}\right)\frac{dx}{\ln x} = -\sum_{k=1}^{\infty} a^k \frac{\ln(1+k)}{k}$$

The change of variables $x = e^{-t}$ gives

$$\int_0^1 \ln\left(\frac{1-ax}{1-a}\right)\frac{dx}{\ln x} = -\int_0^{\infty} \ln\left(\frac{1-ae^{-t}}{1-a}\right)\frac{e^{-t}}{t}\,dt.$$

The expansion

$$\ln(1-ae^{-t}) = -\sum_{k=1}^{\infty} \frac{1}{k} a^k e^{-kt}$$

gives

$$\int_0^1 \ln\left(\frac{1-ax}{1-a}\right)\frac{dx}{\ln x} = \sum_{k=1}^{\infty} \frac{a^k}{k} g(k)$$

where

$$g(k) = \int_0^{\infty} \frac{e^{-kt}-1}{t} e^{-t} dt.$$

Observe that

$$g'(k) = -\int_0^{\infty} e^{-t(k+1)}\,dt = -\frac{1}{k+1}.$$

The initial value $g(0) = 0$, gives $g(k) = -\ln(1+k)$ and the result.