PROOF OF FORMULA 4.222.1

$$\int_0^\infty \ln\left(\frac{a^2+x^2}{b^2+x^2}\right)\,dx = (a-b)\pi.$$

Define

$$f(t) := \int_0^t \ln\left(\frac{a^2+x^2}{b^2+x^2}\right)\,dx.$$

Integration by parts shows that

$$\int_0^t \ln(a^2+x^2)\,dx = t\ln(t^2+a^2) - 2t + 2a^2\int_0^t \frac{dx}{x^2+a^2}.$$

Evaluating this last integral yields

$$\int_0^t \ln(a^2+x^2)\,dx = t\ln(t^2+a^2) - 2t + 2a\tan^{-1}(t/a).$$

It follows that

$$\int_0^t \ln\left(\frac{a^2+x^2}{b^2+x^2}\right)\,dx = tv\ln\left(\frac{t^2+a^2}{t^2+b^2}\right) + 2a\tan^{-1}\frac{t}{a} - 2b\tan^{-1}\frac{t}{b}.$$

Taking the limit as $t \to \infty$ gives the result.