PROOF OF FORMULA 4.227.4

$$\begin{aligned}\int_0^{\pi/4} \ln^n \tan x \, dx &= (-1)^n n! \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{n+1}} \\ &= \frac{1}{2}\left(\frac{\pi}{2}\right)^{n+1} |E_n| \text{ if } n \text{ is even}\end{aligned}$$

Let $v = -\ln \tan x$ to obtain

$$\int_0^{\pi/4} \ln^n \tan x \, dx = (-1)^n \int_0^{\infty} \frac{v^n e^{-v}}{1+e^{-2v}} \, dv.$$

Expand the integrand in a geometric series to obtain

$$\int_0^{\pi/4} \ln^n \tan x \, dx = (-1)^n \sum_{k=0}^{\infty} (-1)^k \int_0^{\infty} v^n e^{-(2k+1)v} \, dv.$$

The change of variables $t = (2k+1)v$ gives the result for general n.

In the case n even use the formula

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2m+1}} = \frac{\pi^{2m+1} |E_{2m}|}{(2m)! 2^{2m+2}}$$

to obtain the expression in terms of the Euler numbers.