PROOF OF FORMULA 4.229.5

$$\int_0^1 \ln(a + \ln x)\,dx = \ln a - e^{-a}\mathrm{Ei}(-a)$$

Let $t = a + \ln x$ to obtain

$$\int_0^1 \ln(a + \ln x)\,dx = e^{-a}\int_{-\infty}^a e^t \ln t\,dt.$$

Integrate by parts to produce

$$\int_0^1 \ln(a + \ln x)\,dx = e^{-a}\left[e^a \ln a - \int_{-\infty}^a \frac{e^t}{t}\,dt\right].$$

That is the result.