PROOF OF FORMULA 4.231.18

$$\int_0^1 \ln x \, \frac{1-x^{n+1}}{(1-x)^2} \, dx = -(n+1)\frac{\pi^2}{6} + \sum_{k=1}^{n} \frac{n-k+1}{k^2}$$

Start with

$$\int_0^1 \ln x \, \frac{1-x^{n+1}}{(1-x)^2} \, dx = \int_0^1 \frac{\ln x}{1-x} \sum_{j=0}^{n} x^j,$$

and expanding $1/(1-x)$ in series yields

$$\int_0^1 \ln x \, \frac{1-x^{n+1}}{(1-x)^2} \, dx = \sum_{j=0}^{n} \sum_{m=0}^{\infty} \int_0^1 x^{j+m} \, \ln x \, dx.$$

The change of variable $u = -\ln x$ gives

$$\int_0^1 \ln x \, \frac{1-x^{n+1}}{(1-x)^2} \, dx = -\sum_{j=0}^{n} \sum_{m=0}^{\infty} \frac{1}{(j+m+1)^2}.$$

The result follows from writing

$$\sum_{m=0}^{\infty} \frac{1}{(j+m+1)^2} = \sum_{m=1}^{\infty} \frac{1}{m^2} - \sum_{m=1}^{j} \frac{1}{m^2}.$$