PROOF OF FORMULA 4.232.1

$$\int_u^v \frac{\ln x\,dx}{(x+u)(x+v)} = \frac{\ln uv}{2(v-u)}\,\ln\frac{(u+v)^2}{4uv}$$

Use the partial fraction decomposition

$$\frac{1}{(x+u)(x+v)} = \frac{1}{v-u}\left(\frac{1}{x+u} - \frac{1}{x+v}\right),$$

to obtain

$$\int_u^v \frac{\ln x\,dx}{(x+u)(x+v)} = \frac{1}{v-u}\left(I_1 - I_2\right),$$

where

$$I_1 = \int_u^v \frac{\ln x\,dx}{x+u} \text{ and } I_2 = \int_u^v \frac{\ln x\,dx}{x+v}.$$

In I_1 let $x = ut$ to produce

$$\begin{aligned} I_1 &= \ln u \int_1^{v/u} \frac{dt}{1+t} + \int_1^{v/u} \frac{\ln t\,dt}{1+t} \\ &= \ln u \ln(1+v/u) - \ln u \ln 2 + \int_1^{v/u} \frac{\ln t\,dt}{1+t}. \end{aligned}$$

A similar change of variable in I_2 produces an analogous result. Let $s = 1/t$ in the second integral to obtain

$$\begin{aligned} I_1 &= \ln u\,(\ln(1+v/u) - \ln 2) + \int_1^{v/u} \frac{\ln t\,dt}{1+t} \\ I_2 &= \ln v\,(\ln 2 - \ln(1+u/v)) + \int_{v/u}^1 \frac{\ln t\,dt}{t(1+t)}. \end{aligned}$$

Therefore,

$$I_1 - I_2 = \ln u\,(\ln(u+v) - \ln u - \ln 2) - \ln v\,(\ln 2 - \ln(u+v) + \ln v) + \int_1^{v/u} \frac{\ln t\,dt}{t}.$$

This last integral is elementary and the result follows from here.