PROOF OF FORMULA 4.232.3

$$\int_0^\infty \frac{\ln x\, dx}{(x+a)(x-1)} = \frac{\pi^2 + \ln^2 a}{2(a+1)}$$

Expand the integrand in partial fractions to obtain

$$\int_0^\infty \frac{\ln x\, dx}{(x+a)(x-1)} = \frac{1}{1+a}\left(\lim_{N\to\infty} I_2(N) - I_3(N)\right)$$

where

$$I_2(N) = \int_0^N \frac{\ln x\, dx}{x-1} \text{ and } I_3(N) = \int_0^N \frac{\ln x\, dx}{x+a}.$$

Split the integral I_2 into two parts: one form 0 to 1 and the second one from 1 to N. Let $t = 1/x$ in the second integral to produce

$$I_2(N) = -\int_0^1 \frac{\ln x\, dx}{1-x} - \int_{1/N}^1 \frac{\ln t}{t} dt - \int_{1/N}^1 \frac{\ln t}{1-t} dt.$$

A similar argument gives

$$I_3(N) = \ln a\, \ln(1+N/a) + \int_0^{N/a} \frac{\ln t\, dt}{1+t}.$$

Thus, to evaluate the original integral and after computing the second integral in I_2, it is required to find the limit as $N\to\infty$ of the expression

$$-\int_0^1 \frac{\ln x\, dx}{1-x} + \frac{1}{2}\ln^2 N - \int_{1/N}^1 \frac{\ln t\, dt}{1-t} - \ln a \ln(1+N/a) - \int_0^{N/a} \frac{\ln t\, dt}{1+t}.$$

Separate the last integral on $0 \le t \le 1$ and $1 \le t \le N$. Then

$$\int_0^\infty \frac{\ln x\, dx}{(x+a)(x-1)} = \frac{1}{1+a}\lim_{N\to\infty}\left(\frac{5\pi^2}{12} + \frac{\ln^2 N}{2} - \ln a \ln(N+a) + \ln^2 a - \int_1^{N/a} \frac{\ln t\, dt}{1+t}\right)$$

Integrate by parts in the last integral to obtain

$$\int_1^{N/a} \frac{\ln t\, dt}{1+t} = \frac{N}{2(a+N)}\ln^2(a/N) - \frac{\pi^2}{12} + o(1).$$

The result now follows from

$$\lim_{N\to\infty} \ln(N+a) - \frac{N}{a+N}\ln N = 0.$$