PROOF OF FORMULA 4.233.1

$$\int_0^1 \frac{\ln x\, dx}{1+x+x^2} = \frac{2}{9}\left(\frac{2\pi^2}{3} - \psi'\left(\frac{1}{3}\right)\right)$$

To evaluate the integral we first compute

$$f(a) = \int_0^1 \frac{t^a \ln t}{1-t}\, dt.$$

Start with

$$\int_0^1 \frac{t^a\, dt}{(1-t)^{1-\epsilon}} = B(a+1,\epsilon) = \frac{\Gamma(a+1)\,\Gamma(\epsilon)}{\Gamma(a+1+\epsilon)}.$$

Differentiate with respect to a to produce

$$\int_0^1 \frac{t^a \ln t\, dt}{(1-t)^{1-\epsilon}} = -\epsilon\Gamma(\epsilon)\frac{\Gamma(a+1)}{\Gamma(a+1+\epsilon)}\left(\frac{\psi(a+1+\epsilon)-\psi(a+1)}{\epsilon}\right).$$

Now let $\epsilon \to 0$ and use $\epsilon\Gamma(\epsilon) \to 1$ to get

$$\int_0^1 \frac{t^a\, \ln t}{1-t}\, dt = -\psi'(a+1).$$

To evaluate the integral, start with

$$\begin{aligned}
\int_0^1 \frac{\ln x\, dx}{1+x+x^2} &= \int_0^1 \frac{1-x}{1-x^3}\, \ln x\, dx \\
&= \int_0^1 \frac{\ln x\, dx}{1-x^3} - \int_0^1 \frac{x \ln x\, dx}{1-x^3} \\
&= \frac{1}{9}\int_0^1 \frac{t^{-2/3}\, \ln t\, dt}{1-t} - \frac{1}{9}\int_0^1 \frac{t^{-1/3} \ln t\, dt}{1-t} \\
&= \frac{1}{9}\left(\psi'\left(\frac{2}{3}\right) - \psi'\left(\frac{1}{3}\right)\right).
\end{aligned}$$

Now use the identity $\psi(1-x) = \psi(x) + \pi\cot\pi x$ and differentiate to get

$$\psi'(1-x) = -\psi'(x) + \frac{\pi^2}{\sin^2 \pi x}.$$

Use this formula to eliminate $\psi'(2/3)$ and obtain the result.