PROOF OF FORMULA 4.234.5

$$\int_0^1 \frac{x^2 \ln x \, dx}{(1-x^2)(1+x^4)} = -\frac{\pi^2}{16(2+\sqrt{2})}$$

Employ the partial fraction decomposition

$$\frac{x^2}{(1-x^2)(1+x^4)} = \frac{1}{4(1-x)} + \frac{1}{4(1+x)} - \frac{1}{2(1+x^4)} + \frac{x^2}{2(1+x^4)},$$

the evaluations

$$\int_0^1 \frac{\ln x \, dx}{1-x} = -\frac{\pi^2}{6} \text{ and } \int_0^1 \frac{\ln x \, dx}{1+x} = -\frac{\pi^2}{12}$$

given as **4.231.2** and **4.231.1** respectively and make the change of variables $t = 1/x$ to obtain

$$\int_0^1 \frac{x^2 \ln x \, dx}{1+x^4} = -\int_1^\infty \frac{\ln t \, dt}{1+t^4}.$$

It follows that

$$\int_0^1 \frac{x^2 \ln x \, dx}{(1-x^2)(1+x^4)} = -\frac{\pi^2}{16} - \frac{1}{2}\int_0^\infty \frac{\ln x \, dx}{1+x^4}.$$

Now let $y = x^4$ to produce

$$\int_0^1 \frac{x^2 \ln x \, dx}{(1-x^2)(1+x^4)} = -\frac{\pi^2}{16} - \frac{1}{32}\int_0^\infty \frac{y^{-3/4} \ln y \, dy}{1+y}.$$

Differentiate the identity

$$\int_0^\infty \frac{y^{t-1} \, dy}{1+y} = \frac{\pi}{\sin \pi t}$$

with respect to the parameter t to obtain

$$\int_0^\infty \frac{y^{t-1} \ln y \, dy}{1+y} = -\frac{\pi^2 \cos \pi t}{\sin^2 \pi t}.$$

The special case $t = \frac{1}{4}$ gives

$$\int_0^\infty \frac{y^{-3/4} \ln y \, dy}{1+y} = -\sqrt{2}\pi^2$$

and the result follows from here.