PROOF OF FORMULA 4.245.1

$$\int_0^1 \frac{x^{4n+1} \ln x \, dx}{\sqrt{1-x^4}} = \frac{(2n-1)!!}{(2n)!!} \frac{\pi}{8} \left(\sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k} - \ln 2 \right)$$

The change of variable $t = x^4$ gives

$$\int_0^1 \frac{x^{4n+1} \ln x \, dx}{\sqrt{1-x^4}} = \frac{1}{16} \int_0^1 t^{n-1/2} (1-t)^{-1/2} \ln t \, dt.$$

In the proof of formula **4.253.1** it was shown that

$$\int_0^1 t^{a-1} (1-t)^{b-1} \ln t \, dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \left[\psi(a) - \psi(a+b) \right].$$

Therefore

$$\int_0^1 \frac{x^{4n+1} \ln x \, dx}{\sqrt{1-x^4}} = \frac{\Gamma(n+1/2)\Gamma(1/2)}{16\Gamma(n+1)} \left[\psi(n+1/2) - \psi(n+1) \right].$$

The values

$$\Gamma(n + \tfrac{1}{2}) = \frac{\sqrt{\pi}}{2^n} (2n-1)!!$$

$$\psi(n + \tfrac{1}{2}) = -\gamma + 2 \left(\sum_{k=1}^n \frac{1}{2k-1} - \ln 2 \right)$$

$$\psi(n+1) = -\gamma + \sum_{k=1}^n \frac{1}{k},$$

give the result.