PROOF OF FORMULA 4.251.1

$$\int_0^\infty \frac{x^{\mu-1}\,\ln x}{b+x}\,dx = \frac{\pi\, b^{\mu-1}}{\sin \pi\mu}\,(\ln b - \pi \cot \pi\mu)$$

Let $x = bt$ to obtain

$$\int_0^\infty \frac{x^{\mu-1}\,\ln x}{b+x}\,dx = b^{\mu-1}\ln b \int_0^\infty \frac{t^{\mu-1}\,dt}{1+t} + b^{\mu-1}\int_0^\infty \frac{t^{\mu-1}\,\ln t}{1+t}\,dt.$$

The integral representation for the beta function

$$B(\alpha,\beta) = \int_0^\infty \frac{t^{\alpha-1}\,dt}{(1+t)^{\alpha+\beta}},$$

gives

$$\int_0^\infty \frac{t^{\mu-1}\,dt}{1+t} = B(\mu, 1-\mu) = \Gamma(\mu)\Gamma(1-\mu) = \frac{\pi}{\sin \pi\mu}.$$

Differentiating with respect to μ yields

$$\int_0^\infty \frac{t^{\mu-1}\,\ln t}{1+t}\,dt = -\pi^2\,\frac{\cos \pi\mu}{\sin^2 \pi\mu}.$$