PROOF OF FORMULA 4.254.6

$$\int_0^1 \frac{x^{q-1}\,\ln x}{1-x^{2q}}\,dx = -\frac{\pi^2}{8q^2}$$

The change of variable $t = x^{2q}$ gives

$$\int_0^1 \frac{x^{q-1}\,\ln x}{1-x^{2q}}\,dx = \frac{1}{4q^2}\int_0^1 \frac{t^{-1/2}\,\ln t}{1-t}\,dt.$$

The identity

$$\int_0^1 \frac{t^{a-1}\,\ln t}{1-t} dt = -\psi'(a)$$

appears in the proof of entry **4.252.3**. Therefore

$$\int_0^1 \frac{x^{q-1}\,\ln x}{1-x^{2q}}\,dx = -\frac{1}{4q^2}\psi'\left(\tfrac{1}{2}\right).$$

The values

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi},\ \Gamma'\left(\tfrac{1}{2}\right) = \sqrt{\pi}\psi\left(\tfrac{1}{2}\right),\ \Gamma''\left(\tfrac{1}{2}\right) = \frac{1}{2}\pi^2\sqrt{\pi} + \sqrt{\pi}\psi'\left(\tfrac{1}{2}\right),$$

to obtain $\psi'\left(\tfrac{1}{2}\right) = \tfrac{1}{2}\pi^2$. Replacing gives the result.