PROOF OF FORMULA 4.255.1

$$\int_0^1 \frac{\ln x}{1+x^{2p}} (1-x^2)x^{p-2}\, dx = -\left(\frac{\pi}{2p} \right)^2 \frac{\sin\left(\frac{\pi}{2p}\right)}{\cos^2\left(\frac{\pi}{2p}\right)}$$

Let $t = x^{2p}$ to obtain

$$\int_0^1 \frac{\ln x}{1+x^{2p}} (1-x^2)x^{p-2}\, dx = \frac{1}{4p^2} \left[\int_0^1 \frac{t^{a-1}\,\ln t}{1+t}\, dt - \int_0^1 \frac{t^{(1-a)-1}\,\ln t}{1+t}\, dt \right]$$

where $a = \frac{1}{2} - \frac{1}{2p}$. Using the function

$$\beta(a) = \int_0^1 \frac{t^{a-1}\, dt}{1+t}$$

the answer becomes

$$\int_0^1 \frac{\ln x}{1+x^{2p}} (1-x^2)x^{p-2}\, dx = \frac{1}{4p^2} \left[\beta'(a) - \beta'(1-a) \right].$$

The result follows from the identities

$$\beta(a) = \frac{1}{2} \left[\psi\left(\frac{1+a}{2} \right) - \psi\left(\frac{a}{2} \right) \right]$$

and

$$\psi'(b) + \psi'(1-b) = \frac{\pi^2}{\sin^2 \pi b}.$$