PROOF OF FORMULA 4.261.13

$$\int_0^1 \frac{x^{2n} \ln^2 x}{1-x^2} \, dx = 2 \sum_{k=n}^{\infty} \frac{1}{(2k+1)^3} = \frac{7}{4}\zeta(3) - 2 \sum_{k=1}^{n} \frac{1}{(2k-1)^3}$$

Expand the integrand in a geometric series to obtain

$$\int_0^1 \frac{x^{2n} \ln^2 x}{1-x^2} \, dx = \sum_{j=0}^{\infty} \int_0^1 x^{2(n+j)} \ln^2 x \, dx.$$

The change of variables $x = e^{-t}$ gives

$$\int_0^1 \frac{x^{2n} \ln^2 x}{1-x^2} \, dx = \sum_{j=0}^{\infty} \int_0^{\infty} t^2 e^{-(2n+2j+1)t} \, dt.$$

Now scale via $s = (2n+2j+1)t$ to produce

$$\int_0^1 \frac{x^{2n} \ln^2 x}{1-x^2} \, dx = 2 \sum_{j=n}^{\infty} \frac{1}{(2k+1)^3}.$$

The identities

$$\sum_{k=n}^{\infty} \frac{1}{(2k+1)^3} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} - \sum_{k=0}^{n-1} \frac{1}{(2k+1)^3},$$

and

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} = \frac{7}{8}\zeta(3)$$

give the result.