PROOF OF FORMULA 4.261.19

$$\int_0^1 \frac{1+(-1)^n x^{n+1}}{(1-x)^2} \ln^2 x \, dx = \frac{3}{2}(n+1)\zeta(3) - 2\sum_{k=1}^{n} (-1)^{k-1} \frac{n+1-k}{k^3}$$

Differentiate

$$\sum_{k=0}^{\infty} x^k = 1/(1-x)$$

to produce

$$\sum_{k=1}^{\infty} k(-1)^{k-1} x^k = 1/(1+x)^2.$$

Thus,

$$\begin{aligned} \frac{1+(-1)^n x^{n+1}}{(1+x)^2} &= \sum_{k=1}^{\infty} (-1)^{k-1} k x^{k-1} + \sum_{k=1}^{\infty} (-1)^{k-1+n} k x^{k+n} \\ &= \sum_{k=1}^{n+1} (-1)^{k-1} k x^{k-1} - (n+1) \sum_{k=n+2}^{\infty} (-1)^k x^{k-1}. \end{aligned}$$

Now use

$$\int_0^1 x^a \ln^2 x \, dx = \int_0^{\infty} t^2 e^{-(a+1)t} \, dt = \frac{2}{(a+1)^3},$$

to obtain

$$\begin{aligned} \int_0^1 \frac{1+(-1)^n x^{n+1}}{(1+x)^2} \ln^2 x \, dx &= \sum_{k=1}^{n+1} (-1)^{k-1} k \int_0^1 x^{k-1} \ln^2 x \, dx - (n+1) \sum_{k=n+2}^{\infty} (-1)^k \int_0^1 x^{k-1} \ln^2 x \, dx \\ &= 2 \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{k^2} - 2(n+1) \sum_{k=n+2}^{\infty} \frac{(-1)^k}{k^3} \end{aligned}$$

and this reduces to the result.