PROOF OF FORMULA 4.267.4

$$\int_0^1 \frac{(1-x)}{(1+x)(1+x^2)} \frac{dx}{\ln x} = -\frac{\ln 2}{2}$$

The function

$$f(a) := \int_0^1 \frac{(1-x)x^a}{(1+x)(1+x^2)} \frac{dx}{\ln x}$$

is evaluated by first using the partial fractions decomposition

$$\frac{(1-x)}{(1+x)(1+x^2)} = \frac{1}{1+x} - \frac{x}{1+x^2}$$

and making the change of variables $x \mapsto \sqrt{x}$ in the second integral to produce

$$f(a) = \int_0^1 \frac{x^a - x^{a/2}}{(1+x)\ln x}\, dx.$$

To evaluate these integrals recall that

$$g(a) := \int_0^1 \frac{x^a\, dx}{1+x} = \beta(a+1) = \frac{1}{2}\psi\left(\frac{a}{2}+1\right) - \frac{1}{2}\psi\left(\frac{a+1}{2}\right)$$

where ψ is the derivative of $\ln\Gamma$. Integrating produces

$$\int_0^1 \frac{x^a - 1}{(1+x)\,\ln x}\, dx = \ln\Gamma\left(\frac{a}{2}+1\right) - \ln\Gamma\left(\frac{a+1}{2}\right) + \frac{1}{2}\ln\pi.$$

Using the duplication formula for the gamma function this can be written as

$$q(a) := \int_0^1 \frac{x^a - 1}{(1+x)\,\ln x}\, dx = 2\ln\Gamma\left(\frac{a}{2}\right) - \ln\Gamma(a) + \ln a + (a-2)\ln 2.$$

The function f is given by

$$f(a) = q(a) - q(a/2) - \frac{\ln 2}{2}$$

and this yields

$$\int_0^1 \frac{(1-x)x^a}{(1+x)(1+x^2)} \frac{dx}{\ln x} = -\ln\Gamma(a) + 3\ln\Gamma\left(\frac{a}{2}\right) - 2\ln\Gamma\left(\frac{a}{4}\right) + \frac{\ln 2}{2}(a+1).$$

The integral requested comes from letting $a \to 0$.