PROOF OF FORMULA 4.271.12

$$\int_0^1 \ln^n x \, \frac{1+x^2}{(1-x^2)^2}\, dx = \frac{1}{2}(2^{2n}-1)\pi^{2n}|B_{2n}|$$

Differentiating the formula for the geometric series gives

$$\frac{1}{(1-x^2)^2} = \sum_{j=1}^{\infty} j x^{2j-2},$$

and replacing yields

$$\int_0^1 \ln^n x \, \frac{1+x^2}{(1-x^2)^2}\, dx = \sum_{j=1}^{\infty} j \int_0^1 x^{2j-2} \ln^{2n} x \, dx + \sum_{j=1}^{\infty} j \int_0^1 x^{2j} \ln^{2n} x \, dx.$$

Now use

$$\int_0^1 x^a \ln^b x \, dx = \frac{(-1)^b \Gamma(b+1)}{(a+1)^{b+1}}$$

to obtain

$$\int_0^1 \ln^n x \, \frac{1+x^2}{(1-x^2)^2}\, dx = (2n)! \sum_{j=1}^{\infty} \frac{1}{(2j-1)^{2n}}.$$

Now use the standard even-odd trick and write

$$\int_0^1 \ln^n x \, \frac{1+x^2}{(1-x^2)^2}\, dx = (2n)!(1-2^{-2n})\zeta(2n).$$

The result now comes from

$$\zeta(2n) = 2^{2n-1}\pi^{2n}\frac{|B_{2n}|}{(2n)!}.$$