PROOF OF FORMULA 4.271.8

$$\int_0^1 \frac{\ln^n x}{1-x^2}\, dx = \frac{2^{2n+1}-1}{2^{2n+1}} (2n)!\, \zeta(2n+1)$$

Expand the integrand as a geometric series to get

$$\int_0^1 \frac{\ln^n x}{1-x^2}\, dx = \sum_{k=0}^{\infty} \int_0^1 x^{2k} \ln^{2n} x \, dx.$$

The change of variable $u = -\ln x$ yields

$$\int_0^1 x^{2k} \ln^{2n} x \, dx = \int_0^{\infty} u^{2n} e^{-(2k+1)u} \, du.$$

Then, $t = (2k+1)u$ gives

$$\int_0^1 x^{2k} \ln^{2n} x \, dx = \frac{1}{(2k+1)^{2n+1}} \int_0^{\infty} t^{2n} e^{-t} \, dt.$$

This last integral is recognized as $\Gamma(2n+1) = (2n)!$. Therefore

$$\int_0^1 \frac{\ln^n x}{1-x^2}\, dx = \frac{1}{(2n)!} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{2n+1}}.$$

The result is now obtained by splitting the series into even and odd indices to produce

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^{2n+1}} = \frac{(2^{n+1}-1)}{2^{2n+1}} \zeta(2n+1).$$