PROOF OF FORMULA 3.197.12

$$\int_0^1 \frac{x^{p-1/2}\,dx}{(1-x)^p\,(1-qx)^p} = \frac{\Gamma(p+\frac{1}{2})\Gamma(1-p)\ \left[(1-\sqrt{q})^{1-2p} - (1+\sqrt{q})^{1-2q}\right]}{\sqrt{\pi q}\,(2p-1)}$$

The integral representation for the hypergeometric function is

$${}_2F_1\left[\alpha,\beta;\gamma;z\right] = \frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}\,dt.$$

Therefore

$$\int_0^1 \frac{x^{p-1/2}\,dx}{(1-x)^p\,(1-qx)^p} = B(p+\tfrac{1}{2},1-p)\ {}_2F_1\left[p,p+\tfrac{1}{2};\tfrac{3}{2};q\right].$$

Introduce the notation $n = 1-2p$ and use formula 9.111 states that

$${}_2F_1\left[\frac{-n+1}{2},\frac{-n+2}{2};\frac{3}{2};z^2\right] = \frac{(1+z)^n-(1-z)^n}{2nz},$$

to get the result.