PROOF OF FORMULA 3.197.7

$$\int_0^\infty x^{\mu-1/2}(x+a)^{-\mu}(x+b)^{-\mu}\,dx = \sqrt{\pi}(\sqrt{a}+\sqrt{b})^{1-2\mu}\frac{\Gamma(\mu-1/2)}{\Gamma(\mu)}$$

Let $x = at$ to obtain

$$\int_0^\infty x^{\mu-1/2}(x+a)^{-\mu}(x+b)^{-\mu}\,dx = \frac{\sqrt{a}}{b^\mu}\int_0^\infty t^{\mu-1/2}(t+1)^{-\mu}(1+at/b)^{-\mu}\,dt.$$

Formula 3.197.5 gives the value of this integral:

$$\int_0^\infty t^{\mu-1/2}(t+1)^{-\mu}(1+at/b)^{-\mu}\,dt = \frac{\sqrt{a}}{b^\mu}B(\mu+\tfrac{1}{2},\mu-\tfrac{1}{2})\,{}_2F_1\left[\mu,\mu+\tfrac{1}{2},2\mu;1-\tfrac{a}{b}\right].$$

To simplify this answer use 9.132.1:

$$\begin{aligned}{}_2F_1\left[\alpha,\beta;\gamma;z\right] &= \frac{(1-z)^{-\alpha}\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}\,{}_2F_1\left[\alpha,\gamma-\beta;\alpha-\beta+1;\frac{1}{1-z}\right]\\ &+ \frac{(1-z)^{-\beta}\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)}\,{}_2F_1\left[\beta,\gamma-\alpha;\beta-\alpha+1;\frac{1}{1-z}\right]\end{aligned}$$

and write $z = 1-\frac{a}{b}$ to obtain

$$\begin{aligned}{}_2F_1\left[\mu,\mu+\tfrac{1}{2},2\mu;z\right] &= \frac{(1-z)^{-\mu}\Gamma(2\mu)\Gamma(1/2)}{\Gamma(\mu+1/2)\Gamma(\mu)}\,{}_2F_1\left[\mu,\mu-\tfrac{1}{2},\tfrac{1}{2};\tfrac{1}{1-z}\right]\\ &+ \frac{(1-z)^{-\mu-1/2}\Gamma(2\mu)\Gamma(-1/2)}{\Gamma(\mu-1/2)\Gamma(\mu)}\,{}_2F_1\left[\mu+\tfrac{1}{2},\mu,\tfrac{3}{2};\tfrac{1}{1-z}\right].\end{aligned}$$

The result now follows from

$${}_2F_1\left[\mu,\mu-\tfrac{1}{2},\tfrac{1}{2};\tfrac{1}{1-z}\right] = \frac{(1+\sqrt{1-z})^{1-2\mu}+(-1+\sqrt{1-z})^{1-2\mu}}{2(1-z)^{1/2-\mu}},$$

that appears as 9.121.2 and

$${}_2F_1\left[\mu,\mu+\tfrac{1}{2},\tfrac{3}{2};\tfrac{1}{1-z}\right] = \frac{(1+\sqrt{1-z})^{1-2\mu}+(-1+\sqrt{1-z})^{1-2\mu}}{2(1-2\mu)(1-z)^{-\mu}},$$

that comes from 9.121.4.