FORMULA 3.222.2

$$\int_0^\infty \frac{x^{\mu-1}\,dx}{x+a} = \frac{\pi}{\sin\pi\mu} \times \begin{cases} a^{\mu-1} & \text{if } a>0 \\ -\cos(\pi\mu)(-a)^{\mu-1} & \text{if } a<0 \end{cases}$$

For $a > 0$ let $x = at$, then

$$\int_0^\infty \frac{x^{\mu-1}\,dx}{x+a} = a^{\mu-1}\int_0^\infty \frac{t^{\mu-1}\,dt}{1+t}.$$

The integral is $B(\mu, 1-\mu) = \pi/\sin\pi\mu$.

For $a < 0$, write $b = -a$ and let $x = bt$. Then

$$\int_0^\infty \frac{x^{\mu-1}\,dx}{x+a} = -b^{\mu-1}\int_0^\infty \frac{t^{\mu-1}\,dx}{1-t}.$$

To evaluate this as a principal value, compute first

$$I(\epsilon) = \int_0^1 \frac{t^{\mu-1}\,dt}{(1-t)^{1-\epsilon}} + \int_1^\infty \frac{t^{\mu-1}\,dt}{(1-t)^{1-\epsilon}}.$$

Let $t \mapsto 1/t$ in the second integral to obtain

$$I(\epsilon) = \int_0^1 t^{\mu-1}(1-t)^{\epsilon-1}\,dt - \int_0^1 t^{-\mu-\epsilon}(1-t)^{\epsilon-1}\,dt.$$

Therefore,

$$I(\epsilon) = B(\mu,\epsilon) - B(\epsilon, 1-\mu-\epsilon).$$

The relation

$$B(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}$$

gives

$$I(\epsilon) = \frac{1}{\Gamma(\mu+\epsilon)\Gamma(1-\mu)}\left(\frac{\Gamma(\mu)\Gamma(1-\mu) - \Gamma(\mu+\epsilon)\Gamma(1-\mu-\epsilon)}{\epsilon}\right).$$

Therefore

$$\int_0^\infty \frac{t^{\mu-1}\,dx}{1-t} = -\frac{\Gamma'(\mu)}{\Gamma(\mu)} + \frac{\Gamma'(1-\mu)}{\Gamma(1-\mu)}.$$

This is $\psi(1-\mu) - \psi(\mu) = \pi\cot(\pi\mu)$.