PROOF OF FORMULA 3.248.4

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)\sqrt{4+3x^2}} = \frac{\pi}{3}$$

This is a special case of 3.248.6:

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)\sqrt{b+ax^2}} = \begin{cases} \frac{2}{\sqrt{b-a}} \tan^{-1}\left(\sqrt{b/a}-1\right) & \text{if } a < b \\ \frac{2}{\sqrt{a}} & \text{if } a = b \\ \frac{1}{\sqrt{a-b}} \ln\left(\frac{\sqrt{a}+\sqrt{a-b}}{\sqrt{a}-\sqrt{a-b}}\right) & \text{if } a > b \end{cases}$$

Define

$$q(a,b) = \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)\sqrt{b+ax^2}}.$$

The change of variables $x = \sqrt{b}t/\sqrt{a}$ gives

$$q(a,b) = 2\sqrt{a}\int_0^{\infty} \frac{dt}{(a+bt^2)\sqrt{1+t^2}}.$$

Now let $t = \tan\theta$ to write the integral in trigonometric form

$$q(a,b) = 2\sqrt{a}\int_0^{\pi/2} \frac{\cos\theta\, d\theta}{a\cos^2\theta + b\sin^2\theta}.$$

The final change of variable $u = \sin\theta$ produces

$$q(a,b) = 2\sqrt{a}\int_0^1 \frac{du}{a+(b-a)u^2}.$$

This is elementary.

The special case $a = 3$ and $b = 4$ gives 3.248.4.