PROOF OF FORMULA 3.411.31

$$\int_0^\infty x \frac{e^{-qx} + e^{(q-p)x}}{1 - e^{-px}} \, dx = \left(\frac{\pi}{p} \text{ cosec } \frac{\pi q}{p} \right)^2$$

The change of variables $t = e^{-px}$ gives

$$\int_0^\infty x \frac{e^{-qx} + e^{(q-p)x}}{1 - e^{-px}} \, dx = -\frac{1}{p^2} \int_0^1 \frac{t^{q/p-1} + t^{-q/p}}{1 - t} \ln t \, dt.$$

Entry 3.231.5 states that

$$\int_0^1 \frac{t^a - t^b}{1 - t} \, dt = \psi(1 + b) - \psi(1 + a).$$

Differentiating with respect to a gives

$$\int_0^1 \frac{t^a \ln t}{1 - t} \, dt = -\psi'(1 + a).$$

Therefore

$$\int_0^\infty x \frac{e^{-qx} + e^{(q-p)x}}{1 - e^{-px}} \, dx = \frac{1}{p^2} \left[\psi' \left(\frac{p}{q} \right) + \psi' \left(1 - \frac{q}{p} \right) \right].$$

The result follows from the identity

$$\psi(1 - z) - \psi(z) = \pi \cot \pi z$$

that yields

$$\psi'(1 - z) + \psi'(z) = \pi^2 \text{ cosec}^2 \pi z$$

obtained by differentiation.