PROOF OF FORMULA 3.475.1

$$\int_0^\infty \left[e^{-x^{2^n}} - \frac{1}{1+x^{2^{n+1}}} \right] \frac{dx}{x} = -\frac{\gamma}{2^n}$$

Consider the more general integral

$$\int_0^\infty \left[e^{-x^a} - \frac{1}{1+x^b} \right] \frac{dx}{x} = -\frac{\gamma}{a}.$$

To prove this, write it as

$$\int_0^\infty \left[e^{-x^a} - \frac{1}{1+x^b} \right] \frac{dx}{x} = \int_0^\infty \left[e^{-x^a} - e^{-x^b} \right] \frac{dx}{x} + \int_0^\infty \left[e^{-x^b} - \frac{1}{1+x^b} \right] \frac{dx}{x}.$$

Entry 3.476.2 states that the first integral is $(a-b)\gamma/ab$. To evaluate the second one, let $t = x^b$ and observe that $\frac{dt}{t} = b\frac{dx}{x}$. This shows that

$$\int_0^\infty \left[e^{-x^b} - \frac{1}{1+x^b} \right] \frac{dx}{x} = \frac{1}{b} \int_0^\infty \left[e^{-t} - \frac{1}{1+t} \right] \frac{dt}{t} = -\frac{\gamma}{b}.$$

It follows that

$$\int_0^\infty \left[e^{-x^a} - \frac{1}{1+x^b} \right] \frac{dx}{x} = \frac{a-b}{ab}\gamma - \frac{\gamma}{b} = -\frac{\gamma}{a},$$

as claimed. The value $a = 2^n$ and $b = 2^{n+1}$ produce the integral requested here.