PROOF OF FORMULA 3.521.1

$$\int_0^\infty \frac{x\,dx}{\sinh x} = \frac{\pi^2}{4}$$

Let $u = e^{-x}$ to obtain

$$\int_0^\infty \frac{x\,dx}{\sinh x} = -2\int_0^1 \frac{\ln u\,du}{1-u^2}.$$

The partial fraction decompositon gives

$$\frac{1}{1-u^2} = \frac{1/2}{1+u} + \frac{1/2}{1-u},$$

gives

$$\int_0^\infty \frac{x\,dx}{\sinh x} = -\int_0^1 \frac{\ln u\,du}{1+u} - \int_0^1 \frac{\ln u\,du}{1-u}.$$

The values

$$\int_0^1 \frac{\ln u\,du}{1+u} = -\frac{\pi^2}{12} \text{ and } \int_0^1 \frac{\ln u\,du}{1-u} = -\frac{\pi^2}{6}$$

are given in formulas 4.231.1 and 4.231.2, respectively. This completes the evaluation.

Note. The table gives the apparently more general result

$$\int_0^\infty \frac{x\,dx}{\sinh ax} = \frac{\pi^2}{4a^2}$$

The change of variables $ax \mapsto x$ shows that the parameter a can be scaled out.