PROOF OF FORMULA 3.546.4

$$\int_0^\infty e^{-bx^2} \cosh^2 ax \, dx = \frac{\sqrt{\pi}}{4\sqrt{b}} \left(\exp\left(\frac{a^2}{b} \right) + 1 \right)$$

Let $t = \sqrt{b}x$ to obtain

$$\int_0^\infty e^{-bx^2} \cosh^2 ax \, dx = \frac{1}{\sqrt{b}} \int_0^\infty e^{-t^2} \cosh^2 ct \, dt.$$

Let J denote the last integral and write $c = a/\sqrt{b}$. Then

$$\begin{aligned} J &= \frac{1}{4} \int_0^\infty e^{-t^2} \left(e^{2ct} + 2 + e^{-2ct} \right) dt \\ &= \frac{1}{4} \int_0^\infty e^{-t^2+2ct} \, dt + \frac{\sqrt{\pi}}{4} + \frac{1}{4} \int_0^\infty e^{-t^2-2ct} \, dt \\ &= \frac{1}{4} \int_0^\infty e^{-t^2+2ct} \, dt + \frac{\sqrt{\pi}}{4} + \frac{1}{4} \int_{-\infty}^0 e^{-t^2+2ct} \, dt \\ &= \frac{1}{4} e^{c^2} \int_{-\infty}^\infty e^{-(t-c)^2} dt + \frac{\sqrt{\pi}}{4} \\ &= \frac{\sqrt{\pi}}{4} \left(e^{c^2} + 1 \right). \end{aligned}$$

This is the result.