PROOF OF FORMULA 3.552.1

$$\int_0^\infty \frac{x^{\mu-1}e^{-\beta x}}{\sinh x}\,dx = 2^{1-\mu}\Gamma(\mu)\zeta\left(\mu, \tfrac{1}{2}(\beta+1)\right)$$

Write the integral as

$$\int_0^\infty \frac{x^{\mu-1}e^{-\beta x}}{\sinh x}\,dx = 2\int_0^\infty \frac{x^{\mu-1}e^{-(\beta+1)x}}{1-e^{-2x}}\,dx.$$

The change of variables $t = 2x$ gives

$$\int_0^\infty \frac{x^{\mu-1}e^{-\beta x}}{\sinh x}\,dx = 2^{1-\mu}\int_0^\infty \frac{t^{\mu-1}e^{-(\beta+1)t/2}}{1-e^{-t}}\,dt.$$

The result now follows from the integral representation of the Hurwitz zeta function

$$\zeta(z,q) = \frac{1}{\Gamma(z)}\int_0^\infty \frac{t^{z-1}e^{-qt}}{1-e^{-t}}\,dt$$

that appears as entry 9.511.1. This is established by expanding the denominator as a geometric series and integrating term by term.