PROOF OF FORMULA 3.552.2

$$\int_0^\infty \frac{x^{2m-1} e^{-ax}}{\sinh ax}\, dx = \frac{1}{2m} |B_{2m}| \left(\frac{\pi}{a} \right)^{2m}$$

The change of variables $t = ax$ gives

$$\int_0^\infty \frac{x^{2m-1} e^{-ax}}{\sinh ax}\, dx = \frac{2}{a^{2m}} \int_0^\infty \frac{t^{2m-1}\, e^{-2t}}{1-e^{-2t}}\, dt.$$

Now let $v = 2t$ to produce

$$\int_0^\infty \frac{x^{2m-1} e^{-ax}}{\sinh ax}\, dx = \frac{1}{2^{2m-1} a^{2m}} \int_0^\infty \frac{v^{2m-1}\, e^{-v}}{1-e^{-v}}\, dv.$$

The integral representation

$$\zeta(z,q) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1}\, e^{-qt}}{1-e^{-t}}\, dt$$

gives

$$\int_0^\infty \frac{x^{2m-1} e^{-ax}}{\sinh ax}\, dx = \frac{1}{2^{2m-1} a^{2m}} \Gamma(2m) \zeta(2m,1).$$

Now observe that

$$\zeta(2m,1) = \sum_{n=0}^\infty \frac{1}{(n+1)^{2m}} = \zeta(2m)$$

and the result follows from the basic identity

$$\zeta(2m) = 2^{2m-1} \pi^{2m} |B_{2m}|/(2m)!$$

and $\Gamma(2m) = (2m-1)!$.