PROOF OF FORMULA 4.224.10

$$\int_0^\pi \ln(1 \pm \sin x)\, dx = -\pi \ln 2 \pm 4G$$

By symmetry

$$\int_0^\pi \ln(1 \pm \sin x)\, dx = 2\int_0^{\pi/2} \ln(1 \pm \sin x)\, dx = 2\int_0^{\pi/2} \ln(1 \pm \cos x)\, dx.$$

The identity $\cos^2 t = \frac{1}{2}(1+\cos t)$ gives

$$\int_0^{\pi/2} \ln(1 \pm \cos x)\, dx = \frac{\pi}{2}\ln 2 + 2\int_0^{\pi/2} \ln\cos\tfrac{x}{2}\, dx.$$

The change of variables $t = x/2$ yields

$$\int_0^{\pi/2} \ln(1 \pm \cos x)\, dx = \frac{\pi}{2}\ln 2 + 4\int_0^{\pi/4} \ln\cos t\, dt.$$

The result now follows from entry 4.224.5 which states

$$\int_0^{\pi/4} \ln\cos t\, dt = -\frac{\pi}{4}\ln 2 + \frac{G}{2}.$$

The case of the minus sign is treated similarly.