PROOF OF FORMULA 4.252.3

$$\int_0^\infty \frac{x^{p-1}\,\ln x}{1-x^2}\,dx = -\left(\frac{\pi}{2\sin(\pi p/2)}\right)^2$$

Let $t = x^2$ to obtain

$$\int_0^\infty \frac{x^{p-1}\,\ln x}{1-x^2}\,dx = \frac{1}{4}\int_0^\infty \frac{t^{p/2-1}\,\ln t}{1-t}\,dt.$$

Split the integral in $0 \leq t \leq 1$ and $1 \leq t < \infty$. Then make the change of variable $s = 1/t$ in the second part. This yields

$$\int_0^\infty \frac{x^{p-1}\,\ln x}{1-x^2}\,dx = \frac{1}{4}\int_0^1 \frac{t^{p/2-1}\,\ln t}{1-t}\,dt + \frac{1}{4}\int_0^1 \frac{t^{-p/2}\,\ln t}{1-t}\,dt.$$

The result now follows from the formula

$$\int_0^1 \frac{t^{a-1}\ln t}{1-t}dt = -\psi'(a)$$

and the relation $\psi'(a) + \psi'(1-a) = \pi^2/\sin^2(\pi a)$. To prove the stated formula, differentiate the beta integral

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt$$

with respect to a to obtain

$$\int_0^1 t^{a-1}(1-t)^{b-1}\ln t\,dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\left(\psi(a) - \psi(a+b)\right).$$

To let $b \to 0$, use $\Gamma(b) = 1/b - \gamma + o(1)$. Write the right-hand side as

$$\frac{\Gamma(a)b\Gamma(b)}{\Gamma(a+b)}\left(\frac{\psi(a)-\psi(a+b)}{b}\right)$$

and then let $b \to 0$.